Representations of the Sergeev superalgebra, queer immanants and Capelli identities

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Plan

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- Idempotents in Sergeev superalgebra

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- Schur Q-polynomials

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Overlapping with independent work by Shuo Li and Lei Shi

Lett. Math. Phys., 2025.

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Permutations $\sigma \in \mathfrak{S}_n \longleftrightarrow$ pairs (P, Q), where Q is barred.

[Worley 1984], [Sagan 1987].

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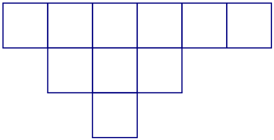
Depict λ by the shifted Young diagram: e.g. for $\lambda = (6, 3, 1)$



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The number g_λ of standard λ-tableaux is found by the Schur formula [1911] ([Morris 1965] gave a hook-length formula):

$$g_\lambda = \frac{n!}{\lambda_1! \ldots \lambda_\ell!} \prod_{1 \leqslant i < j \leqslant \ell} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$

The barred tableaux are obtained by allowing any non-diagonal entry to occur with a bar on it:

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & \bar{4} & \bar{5} & 8 & \overline{10} \\ \hline \multicolumn{1}{c|}{} & 3 & \bar{6} & 9 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 7 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array} .$$

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We will use the barred tableaux to construct representations of the Sergeev superalgebra.

Spin symmetric group algebra

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This is the superalgebra $\mathbb{C}\mathfrak{S}_n^-$ generated by odd elements

$t_1, \ldots, t_{n-1}$ subject to the relations

$$t_a^2 = 1, \qquad t_a t_{a+1} t_a = t_{a+1} t_a t_{a+1}, \qquad t_a t_b = -t_b t_a, \quad |a-b| > 1.$$

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The transpositions are defined by

$$t_{ab} = (-1)^{b-a-1}\, t_{b-1} \dots t_{a+1} t_a t_{a+1} \dots t_{b-1}, \qquad a < b,$$

and $t_{ba} = -t_{ab}$.

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The odd Jucys–Murphy elements are

$$m_1 = 0, \qquad m_a = t_{1a} + \dots + t_{a-1,a}, \qquad a = 2, \dots, n.$$

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Also, $\mathcal{S}_n = \mathbb{C}\mathfrak{S}_n \ltimes \mathcal{C}l_n$ with

$$s_a c_a = c_{a+1} s_a, \quad s_a c_{a+1} = c_a s_a, \quad s_a c_b = c_b s_a, \quad b \neq a, a+1,$$

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where

$$s_a = \frac{1}{\sqrt{2}} t_a (c_{a+1} - c_a).$$

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The dimensions of the simple modules over $\mathcal{S}_n$ and $\mathbb{C}\mathfrak{S}_n^-$:

$$\dim U^\lambda = 2^{n-\lfloor \frac{\ell(\lambda)}{2} \rfloor} g_\lambda \qquad \text{and} \qquad \dim V^\lambda = 2^{\lceil \frac{n-\ell(\lambda)}{2} \rceil} g_\lambda.$$

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Both superalgebras are semisimple; the Wedderburn decompositions yield the Schur formula.

Given a standard barred tableaux $\mathcal{U}$, the signed content $\kappa_{\mathsf{a}}(\mathcal{U})$

of any barred or unbarred entry a of $\mathcal{U}$ is

$$\kappa_{\mathsf{a}}(\mathcal{U}) = \begin{cases} \sqrt{\sigma_a(\sigma_a+1)} & \text{if } \mathsf{a}=a \text{ is unbarred,} \\ -\sqrt{\sigma_a(\sigma_a+1)} & \text{if } \mathsf{a}=\bar{a} \text{ is barred,} \end{cases}$$

$\sigma_a = j - i$ is the content of the box (i, j) of λ occupied by a or $\bar{a}$.

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$\sigma_a = j - i$ is the content of the box (i, j) of λ occupied by a or $\bar{a}$.

The even Jucys–Murphy elements in $\mathcal{S}_n$ are [Nazarov 1997]:

$$x_b = \sqrt{2}\, m_b\, c_b = \sum_{a=1}^{b-1} (ab)(1 + c_a c_b), \qquad b = 1, \ldots, n.$$

Note that $x_b^2 = 2m_b^2$ and the x_b pairwise commute.

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$$\mathcal{U} \quad = \quad \begin{array}{|c|c|c|c|c|c|}
\hline
1 & 2 & \overline{4} & \overline{5} & 8 & \overline{10} \\
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Delete the box with $\overline{10}$ to get the tableau

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Addable boxes with signed contents: $\sqrt{30}$, $\quad \pm\sqrt{12}$, $\quad \pm\sqrt{2}$.

For any standard barred tableau $\mathcal{U}$ introduce the element $e_{\mathcal{U}}$ of $\mathcal{S}_n$ by induction, setting $e_{\boxed{1}} = 1$ and

$$e_{\mathcal{U}} = e_{\mathcal{V}} \, \frac{(x_n - b_1) \dots (x_n - b_p)}{(\kappa - b_1) \dots (\kappa - b_p)},$$

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where $\mathcal{V}$ is the barred tableau obtained from $\mathcal{U}$ by removing the box α occupied by n (resp. $\bar{n}$); cf. [Jucys 1971], [Murphy 1981].

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where $\mathcal{V}$ is the barred tableau obtained from $\mathcal{U}$ by removing the box α occupied by n (resp. $\bar{n}$); cf. [Jucys 1971], [Murphy 1981].

Here $b_1, \dots, b_p$ are the signed contents in all addable boxes of $\mathrm{sh}(\mathcal{V})$ (barred and unbarred), except for the entry n (resp. $\bar{n}$), while κ is the signed content of the entry n (resp. $\bar{n}$).

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The respective elements $e_{\mathcal{U}} \in \mathcal{S}_3$ are

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we have

$$e_{\mathcal{U}} = \frac{\sqrt{2} - x_2}{2\sqrt{2}} \cdot \frac{(x_3 - \sqrt{6})(x_3 + \sqrt{6})}{-6} = \frac{\sqrt{2} - x_2}{2\sqrt{2}} \cdot \frac{6 - x_3^2}{6}.$$

Theorem. All elements $e_{\mathcal{U}}$ are idempotents in $\mathcal{S}_n$. They are pairwise orthogonal and form a decomposition of the identity:

$$e_{\mathcal{U}}\, e_{\mathcal{V}} = \delta_{\mathcal{U}\mathcal{V}}\, e_{\mathcal{V}}, \qquad 1 = \sum_{\lambda \Vdash n} \sum_{\mathrm{sh}(\mathcal{U})=\lambda} e_{\mathcal{U}}.$$

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Moreover,

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Proof. Use the faithful $\mathcal{S}_n$-module

$$\widehat{U} = \bigoplus_{\lambda \Vdash n} \widehat{U}^{\lambda}.$$

Here

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The generators of $\mathcal{S}_n$ act by

$$s_a v_{\mathcal{T}} = \left(\frac{1}{\kappa_{a+1}(\mathcal{T}) - \kappa_a(\mathcal{T})} + \frac{c_a c_{a+1}}{\kappa_{a+1}(\mathcal{T}) + \kappa_a(\mathcal{T})} \right) v_{\mathcal{T}} + \mathcal{Y}_a(\mathcal{T})\, v_{s_a \mathcal{T}},$$

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$$\mathcal{Y}_a(\mathcal{T}) = \sqrt{1 - \frac{1}{\big(\kappa_a(\mathcal{T}) - \kappa_{a+1}(\mathcal{T})\big)^2} - \frac{1}{\big(\kappa_a(\mathcal{T}) + \kappa_{a+1}(\mathcal{T})\big)^2}}.$$

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Corollary. We have the direct sum decomposition

$$\mathcal{S}_n = \bigoplus_{\lambda \Vdash n} \bigoplus_{\mathrm{sh}(\mathcal{U})=\lambda} \mathcal{S}_n \, e_{\mathcal{U}},$$

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Moreover,

$$\mathcal{S}_n e_{\mathcal{U}} \cong \widehat{U}^{\lambda} \qquad \text{and} \qquad \widehat{U}^{\lambda} \cong \underbrace{U^{\lambda} \oplus \cdots \oplus U^{\lambda}}_{2^{\lfloor \frac{\ell(\lambda)}{2} \rfloor}}.$$

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Consider commuting actions of $\mathfrak{S}_n$ and $\mathfrak{gl}_N$ on the space

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Let $\mathcal{U}$ be a standard tableau of shape $\lambda \vdash n$ with $\ell(\lambda) \leqslant N$.

Primitive idempotent $e_{\mathcal{U}}$ for $\mathfrak{S}_n$ acts as an operator $\mathcal{E}_{\mathcal{U}}$.

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$$\operatorname{tr} \mathcal{E}_{\mathcal{U}} Y_1 \ldots Y_n = s_\lambda(y_1, \ldots, y_N),$$

where $Y = \operatorname{diag}(y_1, \ldots, y_N)$ and $Y_a = 1^{\otimes(a-1)} \otimes Y \otimes 1^{\otimes(n-a)}$.

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Consider the $\mathbb{Z}_2$-graded vector space $\mathbb{C}^{N|N}$ with the canonical basis $e_{-N}, \dots, e_{-1}, e_1, \dots, e_N$ with $p(e_i) = \bar{\imath} \mod 2$, where

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The Sergeev superalgebra $\mathcal{S}_n = \mathbb{C}\mathfrak{S}_n \ltimes \mathcal{C}l_n$ acts on the space

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by $\qquad c_a \mapsto J_a \qquad$ and $\qquad (a, b) \mapsto P_{ab},$

Sergeev duality

Consider the $\mathbb{Z}_2$-graded vector space $\mathbb{C}^{N|N}$ with the canonical basis $e_{-N}, \ldots, e_{-1}, e_1, \ldots, e_N$ with $p(e_i) = \bar{\imath} \mod 2$, where

$$\bar{\imath} = 1 \quad \text{for} \quad i < 0 \qquad \text{and} \qquad \bar{\imath} = 0 \quad \text{for} \quad i > 0.$$

The Sergeev superalgebra $\mathcal{S}_n = \mathbb{C}\mathfrak{S}_n \ltimes \mathcal{C}l_n$ acts on the space

$$\underbrace{\mathbb{C}^{N|N} \otimes \ldots \otimes \mathbb{C}^{N|N}}_{n}$$

by $\quad c_a \mapsto J_a \quad$ and $\quad (a, b) \mapsto P_{ab},$

with

$$J = \sum_{i=-N}^{N} E_{i,-i}(-1)^{\bar{\imath}} \in \mathrm{End}\,\mathbb{C}^{N|N}.$$

The superalgebra $Q_N \subset \operatorname{End} \mathbb{C}^{N|N}$ has the basis

$$e_{kl} = E_{kl} + E_{-k,-l}, \qquad f_{kl} = E_{k,-l} + E_{-k,l}, \qquad 1 \leqslant k, l \leqslant N,$$

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[Sergeev, 1985].

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and $Q_\lambda(y_1, \dots, y_N)$ is the Schur Q-polynomial.

Factorial Schur Q-polynomials

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Use the alphabet $1' < 1 < 2' < 2 < \cdots < N' < N$ and set

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$1'$	1	$2'$	2	2	2
	2	$3'$	3		
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$$Q_\lambda^+(y) = \sum_{\mathrm{sh}(\mathcal{T})=\lambda} \prod_{\alpha\in\lambda} \big(y_{\mathcal{T}(\alpha)} + \operatorname{sgn}(\mathcal{T}(\alpha))\sigma(\alpha)\big),$$

where $\sigma(\alpha) = j - i$ is the content of $\alpha = (i,j)$ [Ivanov, 2005].

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Vanishing property:

$$Q_\lambda^+(-\mu) = 0$$

for all strict μ with $\ell(\mu) \leqslant N$ and $|\mu| < |\lambda|$.

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Characterization property [Ivanov, 2005]:

If the top degree component of a polynomial $P(y) \in \Gamma_N$ coincides with $Q_\lambda(y)$ for some λ with $\ell(\lambda) \leqslant N$, and $P(-\mu) = 0$ for all strict μ with $\ell(\mu) \leqslant N$ and $|\mu| < |\lambda|$, then $P(y) = Q_\lambda^+(y)$.

Differential operators

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Define an action of $\mathfrak{q}_N$ in $\mathcal{P}$ by

$$e_{kl} \mapsto - \sum_{a=-M}^{M} x_{al} \partial_{ka},$$

$$f_{kl} \mapsto i \sum_{a=-M}^{M} x_{al} \partial_{k,-a}.$$

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We thus get a homomorphism $\mathrm{U}(\mathfrak{q}_N) \to \mathcal{P}\mathcal{D}$.

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Combine the variables and derivations into matrices X and D:

$$X = \sum_{a=1}^{M} \sum_{k=1}^{N} \left(e_{ka} \otimes x_{ak} - i f_{ka} \otimes x_{-a,k} \right) \in \mathrm{Hom}(\mathbb{C}^M, \mathbb{C}^{N|N}) \otimes \mathcal{PD}$$

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Introduce the odd element

$$G = \sum_{k,l=1}^{N} \big(e_{kl} \otimes f_{lk} - f_{kl} \otimes e_{lk}\big) \in Q_N \otimes \mathrm{U}(\mathfrak{q}_N).$$

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and for $b = 1, \ldots, n$ set

$$M^{(b)} = T_{1b} + \cdots + T_{b-1,b} \in (Q_N)^{\otimes n}.$$

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This is the image of the odd Jucys–Murphy element.

Theorem. Under the action $G \mapsto XD$, we have

$$(G_1 + M^{(1)}) \dots (G_n + M^{(n)}) \mapsto X_1 \dots X_n D_1 \dots D_n.$$

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Proof. Simple induction argument.

Quantum immanants

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The quantum immanant associated with $\lambda \Vdash n$ is defined by

$$\mathbb{S}_\lambda = \operatorname{str} \mathcal{E}_{\mathcal{U}} \big(F_1 + \kappa_1(\mathcal{U})\big) \dots \big(F_n + \kappa_{\mathsf{n}}(\mathcal{U})\big) \in \mathrm{U}(\mathfrak{q}_N),$$

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Moreover, it depends only on λ and does not depend on $\mathcal{U}$.

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On any highest vector $\xi \in L(y)$ we have

$$F_{ii}\, \xi = y_i\, \xi, \qquad i = 1, \dots, N,$$

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The scalar is a supersymmetric polynomial denoted by $\chi(\mathbb{S}_\lambda)$.

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Proof. The top degree component of $\chi(\mathbb{S}_\lambda)$ is the supertrace

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Now use the characterization of $Q^+_\lambda(y)$; the vanishing property is implied by the Capelli identity.

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Relation to earlier work

[Nazarov, 1997]:

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They introduced elements z_λ of $\mathrm{Z}(\mathfrak{q}_N)$ and found the Harish-Chandra images $\chi(z_\lambda)$. We can conclude that

$$z_\lambda = \frac{(-1)^n \, g_\lambda}{2^{\ell(\lambda)} \, n!} \, \mathbb{S}_\lambda.$$