NOT ON EXAM

Differentiation of functions of 2 variables

First recall that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$ if there is a number $f'(a) \in \mathbb{R}$ so that

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

<u>Definition</u> A function $T : \mathbb{R} \to \mathbb{R}$ is <u>linear</u> if for all $x, y \in \mathbb{R}$ and all scalars $\lambda, \mu \in \mathbb{R}$

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y).$$

<u>Exercise</u> $T : \mathbb{R} \to \mathbb{R}$ is linear if and only if $T(x) = mx$ for some $m \in \mathbb{R}$.

Define a linear function $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = f'(a)x$. So $T(h) = f'(a)h$.
Note $y = T(x)$ is line through $(0,0)$, slope $f'(a)$.

Then (2)

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a) \iff \lim_{h\to 0} \frac{f(a+h)-f(a)-T(h)}{h} = 0.$$

So we may redefine f to be differentiable at $a \in \mathbb{R}$ if there is a linear function $T: \mathbb{R} \to \mathbb{R}$ so that

$$\lim_{h\to 0} \frac{f(a+h)-f(a)-T(h)}{h} = 0.$$

<u>Exercise</u> If such a linear function exists, it is unique (see Taylor polynomials Week 8 lecture 2).

What this is saying is that the function $x \mapsto T(x-a)+f(a)$ is a good approximation to f near a. But $T(x-a)+f(a) = f'(a)(x-a)+f(a)$, which is the tangent line at $x=a$!

Now consider $f: \mathbb{R}^2 \to \mathbb{R}$.

<u>Definition</u> A function $T: \mathbb{R}^2 \to \mathbb{R}$ is <u>linear</u> if for all vectors $\underline{x}, \underline{y} \in \mathbb{R}^2$ and all scalars $\lambda, \mu \in \mathbb{R}$,

$$T(\lambda \underset{\sim}{x} + \mu \underset{\sim}{y}) = \lambda T(\underset{\sim}{x}) + \mu T(\underset{\sim}{y}).$$

<u>Exercise</u> (linear algebra)

1. If $A = (a_{11} \;\; a_{12})$ is a 1×2 matrix, then the function $T : \mathbb{R}^2 \to \mathbb{R}$ given by

$$T(\underset{\sim}{x}) = A\underset{\sim}{x} \text{ i.e. } T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (a_1 \;\; a_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a_1x_1 + a_2x_2$$

is linear.

2. For any linear function $T : \mathbb{R}^2 \to \mathbb{R}$, there is a unique 1×2 matrix A so that

$$T(\underset{\sim}{x}) = A\underset{\sim}{x} \quad \text{for all } \underset{\sim}{x} \in \mathbb{R}^2.$$

3. Formulate a definition of linearity for functions $\mathbb{R}^m \to \mathbb{R}^n$ and show that they are in bijection with the set of $n \times m$ matrices.

<u>Definition</u> A function $f : \mathbb{R}^2 \to \mathbb{R}$ is <u>differentiable at $(a,b) \in \mathbb{R}^2$</u> if there is a linear map $T : \mathbb{R}^2 \to \mathbb{R}$ so that

$$\lim_{(h_1,h_2)\to(0,0)} \frac{f(a+h_1, b+h_2) - f(a,b) - T(h_1,h_2)}{|(h_1,h_2)|} = 0$$

Theorems

1. If such a linear map exists, it is unique.
2. If f_x and f_y exist and are continuous on a disc containing (a,b), then f is differentiable at (a,b).
3. If f is differentiable at (a,b) then T is given by the matrix of partial derivatives at (a,b)

   i.e. $T(\underset{\sim}{x}) = \begin{pmatrix} f_x(a,b) & f_y(a,b) \end{pmatrix} \underset{\sim}{x}$

   The <u>derivative of f at (a,b)</u> is the 1×2 matrix

   $$\begin{pmatrix} f_x(a,b) & f_y(a,b) \end{pmatrix}.$$
4. This definition is saying that

   $(x,y) \longmapsto T(x-a, y-b) + f(a,b)$ is a good approximation to f near (a,b).

   But $T(x-a, y-b) + f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$

   which is the tangent plane!

5. There are functions f such that f_x and f_y exist but f is <u>not</u> differentiable

e.g. $f(x,y) = \begin{cases} 1 & \text{if } x=0 \text{ or } y=0 \\ 0 & \text{otherwise} \end{cases}$

is not differentiable at $(0,0)$ even though $f_x(0,0)=0$ and $f_y(0,0)=0$.

6. If f is differentiable at (a,b) then f is continuous at (a,b).