Week 13 – Lectures 1 and 2

Revision – this is not comprehensive, and you should make your own summary!

Functions

$$f: A \longrightarrow B$$

A = domain, B = codomain

The range of f is $f(A) \subseteq B$.

$f(A) = \{f(a) \mid a \in A\}$ = image of A

Examples

1. Polynomials

- 1 variable e.g. $f: \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^3+5x+17$ or $f(x) = x^3+5x+17$
- 2 variables e.g. $f: \mathbb{R}^2 \to \mathbb{R}$ given by $(x,y) \mapsto x^2y^2+3xy^4$ or $f(x,y) = x^2y^2+3xy^4$
- complex polynomials e.g. $f: \mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^2+2$ or $f(z) = z^2+2$.

• Fundamental Theorem of Algebra: a complex polynomial has at least one root

• a complex polynomial p of degree n factors as $\alpha(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)$, $\alpha, \alpha_i \in \mathbb{C}$

i.e. p has n roots over $\mathbb{C}$ (not nec. all distinct)

• if a complex polynomial has real coefficients then its complex roots occur in complex conjugate pairs

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2. Rational functions: quotients of polynomials

- 1 variable: domain of $\frac{p(x)}{q(x)}$ is $\{x \in \mathbb{R} \mid q(x) \neq 0\}$

- 2 variables: domain of $\frac{p(x,y)}{q(x,y)}$ is $\{(x,y) \in \mathbb{R}^2 \mid q(x,y) \neq 0\}$

- complex: domain of $\frac{p(z)}{q(z)}$ is $\{z \in \mathbb{C} \mid q(z) \neq 0\}$

3. Trig. functions

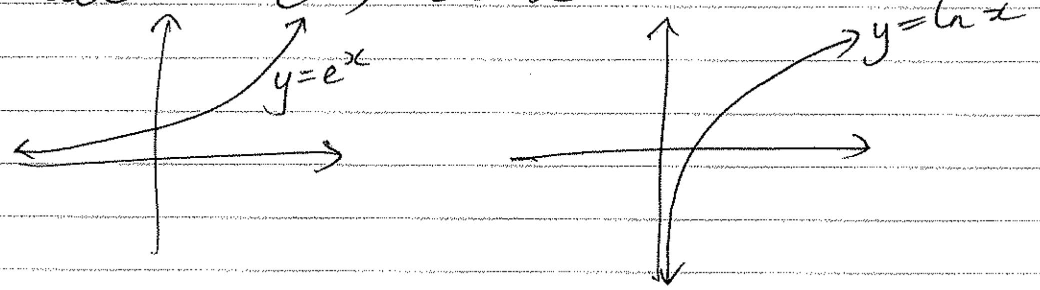
- real: $\sin : \mathbb{R} \to \mathbb{R}$, $\cos : \mathbb{R} \to \mathbb{R}$ have range $[-1, 1]$

  $\tan x = \frac{\sin x}{\cos x}$ has range $\mathbb{R}$

- complex: see assignment

4. exponential and logarithm

- real: e^x, $\ln x$



- complex exponential (we did not discuss complex ln)

  $\exp z$ or e^z

  If $z = x+iy$ then $e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$

  Understand by considering image of regions A in the complex plane $\mathbb{C}$.

5. hyperbolic trig functions

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

	$\cosh$	$\sinh$
domain	$\mathbb{R}$	$\mathbb{R}$
range	$[1, \infty)$	$(-\infty, \infty) = \mathbb{R}$

6. <u>Modulus</u> $z \mapsto |z|$ is a function $\mathbb{C} \to \overset{\mathbb{R}}{\cancel{[0,\infty)}}$
with range $[0, \infty)$

If $z, w \in \mathbb{C}$ then $|z-w|$ is distance between z and w.

<u>Principal argument</u> $z \mapsto \operatorname{Arg} z$ is a function
$\mathbb{C} \setminus \{0\} \longrightarrow \mathbb{R}$ with range $[-\pi, \pi)$.

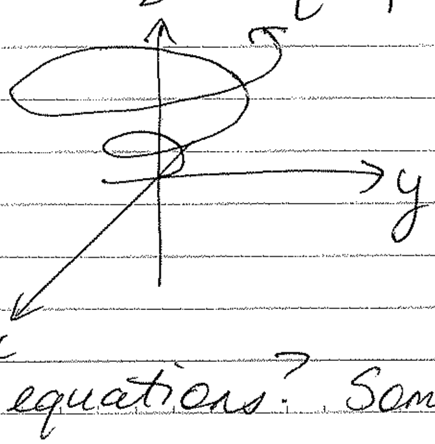
Polar form
$re^{i\theta}$ has modulus r, principal argument $\theta + 2k\pi$ for some $k \in \mathbb{Z}$.

<u>Complex conjugate</u> $z \longmapsto \bar{z}$ is a function $\mathbb{C} \to \mathbb{C}$ with range $\mathbb{C}$.

7. Curves
- in $\mathbb{R}^2$ e.g. $\phi : [0, \infty) \longrightarrow \mathbb{R}^2$
$t \longmapsto (t\cos t, t \sin t)$
spiral going out

- in $\mathbb{R}^3$ e.g. $\phi : [0, \infty) \longrightarrow \mathbb{R}^3$
$t \longmapsto (t\cos t, t\sin t, t^2)$

$z = x^2 + y^2$



e.g.

Cartesian equations? Sometimes exist.

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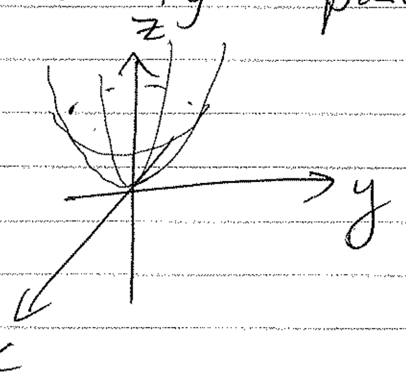
8. A <u>surface in $\mathbb{R}^3$</u> can be given by $z = f(x,y)$

e.g. $z = x^2 + y^2$ paraboloid



Understand via <u>level curves</u> $f(x,y) = k$

Not all surfaces are of the form $z = x^2 + y^2$
e.g. unit sphere $x^2 + y^2 + z^2 = 1$.

<u>Injective / Surjective / Bijective</u>

A function $f: A \to B$ is <u>injective</u> if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2) \implies a_1 = a_2$.

A function $f: A \to B$ is <u>surjective</u> if for all $b \in B$, there is at least one $a \in A$ so that $f(a) = b$.

A function $f: A \to B$ is <u>bijective</u> if ~~for~~ it is both injective and surjective. Equivalently, for every $b \in B$, there is exactly one $a \in A$ so that $f(a) = b$.

Note: vertical / horizontal line tests only useful in <u>some</u> circumstances.

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Examples

1. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = z^n$, n a positive integer.

Then for $n \geq 2$, f is <u>not</u> injective: for any $w \neq 0$, $z^n = w$ has n distinct solutions.

To find them: put $z = re^{i\theta}$ $w = se^{i\phi}$
then $z^n = w \Rightarrow r^n = s$ and $n\theta = \phi + 2k\pi$ $k \in \mathbb{Z}$
$\Rightarrow r = \sqrt[n]{s}$ and $\theta = \frac{\phi}{n} + \frac{2k\pi}{n}$ $k \in \mathbb{Z}$

2. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = e^z$.

Then f is <u>not</u> injective since e.g. $f(0) = f(2\pi i)$.

Also f is <u>not</u> surjective since there is no $z \in \mathbb{C}$ so that $f(z) = 0$.

However $g : A \to B$ given by $g(z) = e^z$ is a bijection if $A = \{ z \in \mathbb{C} \mid 0 \leq \operatorname{Im}(z) < 2\pi \}$
$B = \mathbb{C} \setminus \{0\}$.

3. Let $f : [0, \infty) \to [1, \infty)$ be given by $f(x) = \cosh x$.

Then f is a bijection.

Limits

- 1 variable:
$\lim_{x \to a} f(x) = l$ if for all $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < |x - a| < \delta$, $|f(x) - l| < \varepsilon$.
- 2 variables:
$\lim_{(x,y) \to (a,b)} f(x,y) = l$ if for all $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < |(x,y) - (a,b)| < \delta$, $|f(x) - l| < \varepsilon$

(6) To prove statements about limits, you MUST know these definitions!

To compute limits, can use

Limit Laws

- If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$ then
  $\lim_{x\to a}(f(x) \pm g(x)) = l \pm m$
  $\lim_{x\to a}(f(x)g(x)) = lm$
  $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{l}{m}$ if $m \neq 0$
- Squeeze Law
- Substitution Law

also for 2 variables

- L'Hôpital's rule (uses differentiation, 1 variable only)

key difference

- 1 variable: $\lim_{x\to a} f(x) = l \iff \lim_{x\to a^+} f(x) = l$ and $\lim_{x\to a^-} f(x) = l$
- 2 variables: infinitely many ways that (x,y) can approach (a,b), $\lim_{(x,y)\to(a,b)} f(x,y) = l$ only if $f(x,y) \to l$ along all of these paths.

Continuity

- 1 variable / 2 variables: similar definitions
  $\lim_{x\to a} f(x) = f(a)$, $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.
- continuous functions: polynomials, rational functions, trig functions, exponential + logarithm

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Important theorems about continuous functions on closed and bounded intervals $[a,b]$

- Intermediate Value Theorem
  - uses Least Upper Bound Property
- Extreme Value Theorem — also for 2 variables
  - uses theorem that a continuous function on $[a,b]$ is bounded

Differentiation

· 1 variable

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \rightarrow \text{tangent line}$$

· 2 variables

- partial derivatives

$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

$\rightarrow$ tangent plane, linear approximation, gradient ∇f

- implicit differentiation Implicit Function Theorem: if $f_y(a,b) \neq 0$ then $f(x,y)=k$ defines y implicitly as a function of x, and $\frac{dy}{dx} = \frac{-f_x}{f_y}$.

- directional derivative $\underset{\sim}{u}$ unit vector

$$D_{\underset{\sim}{u}} f(x,y) = \lim_{h\to 0^+} \frac{f(x+hu_1, y+hu_2)-f(x,y)}{h}$$

$$= \underset{\sim}{u} \cdot (\underset{\sim}{\nabla} f(x,y))$$

Use for steepest slope, level directions, ...

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Rules for differentiation

- Single variable and partial derivatives:
  - sums, differences, product + quotient rules

- Chain rule
  - 1 variable $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
  - 2 variables - 2 versions

    If $z=f(x,y)$ and $x=g(t)$, $y=h(t)$

    $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

    If $z=f(x,y)$ and $x=g(s,t)$, $y=h(s,t)$

    $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

- Derivative of the inverse (1 variable only)

  $$(f^{-1})'(f(a)) = \frac{1}{f'(a)} \qquad \text{if } f'(a) \neq 0$$

Using derivatives to find max/min

- 1 variable: if f differentiable on the interval (a,b) and f has a local max or min at $c \in (a,b)$ then $f'(c)=0$
- 2 variables: if f_x and f_y exist and are continuous near the point (a,b) and f has a local max or min at (a,b) then $f_x(a,b)=0$ and $f_y(a,b)=0$.

Important theorems about differentiable functions

of 1 variable

- Rolle's Theorem

- Mean Value Theorem + its corollaries

- Cauchy Mean Value Theorem

Taylor Polynomials

- know the formula
- construction: the unique poly. of degree $\leq n$ which has the same derivatives as f at a, up to degree n.
- error estimate: Lagrange's form of remainder
- property: the unique poly. of degree $\leq n$ s.t.

$$\lim_{x \to a} \frac{f(x) - T_n(x)}{(x-a)^n} = 0.$$

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