Week 10 Lecture 1

Recap:

1. Curves

in $\mathbb{R}^2$ e.g. $t \mapsto (\overset{x}{\cos t}, \overset{y}{\sin t})$ circle

in $\mathbb{R}^3$ e.g. $t \mapsto (\overset{x}{\cos t}, \overset{y}{\sin t}, \overset{z}{t})$ helix

2. Surfaces

- given by equations in x, y and z

e.g. $x^2 + y^2 + z^2 = 1$ sphere

$ax + by + cz = d$ plane

- given as graphs $z = f(x,y)$

where $f : D \longrightarrow \mathbb{R}$ where domain D is subset of $\mathbb{R}^2$

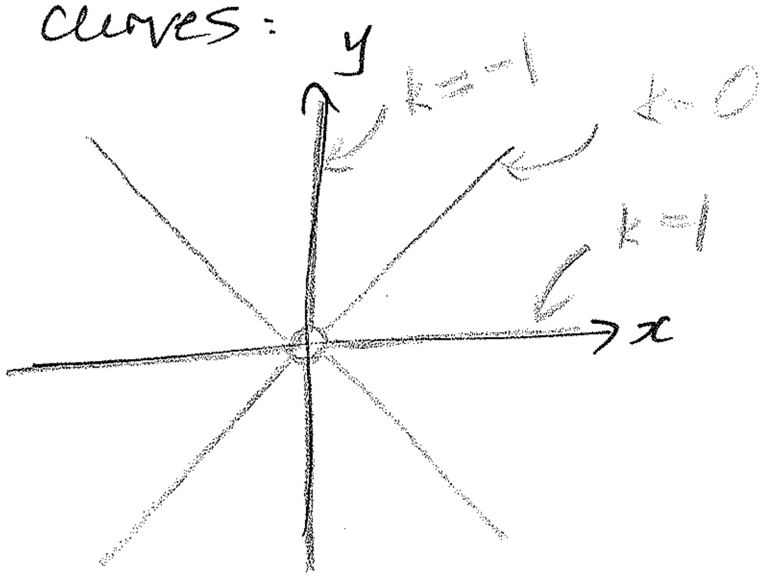
To understand graphs, can look at
- domain and range of f
- level curves i.e. (x,y) such that $f(x,y) = k$
- where the graph intersects
  the yz-plane i.e. $x = 0$
  the xz-plane i.e. $y = 0$

(level curve at height 0 is intersection with xy-plane) e.g. let $h(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$

domain of h is $\underbrace{\mathbb{R}^2 \setminus \{(0,0)\}}_{\text{all of } \mathbb{R}^2 \text{ except } (0,0)}$

range of h is $[-1,1]$
use polar coords $x = r\cos\theta$, $y = r\sin\theta$ then
$h(x,y) = \cos 2\theta$

level curves:



Limits and continuity for functions of two variables

$f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$.

Let $(a,b) \in D$.

Definition

The limit of $f(x,y)$ as (x,y) approaches (a,b) exists and equals ℓ, denoted

$$\lim_{(x,y)\to(a,b)} f(x,y) = \ell$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever

$$0 < \underbrace{|(x,y)-(a,b)|}_{\text{distance between } (x,y) \text{ and } (a,b)} < \delta$$

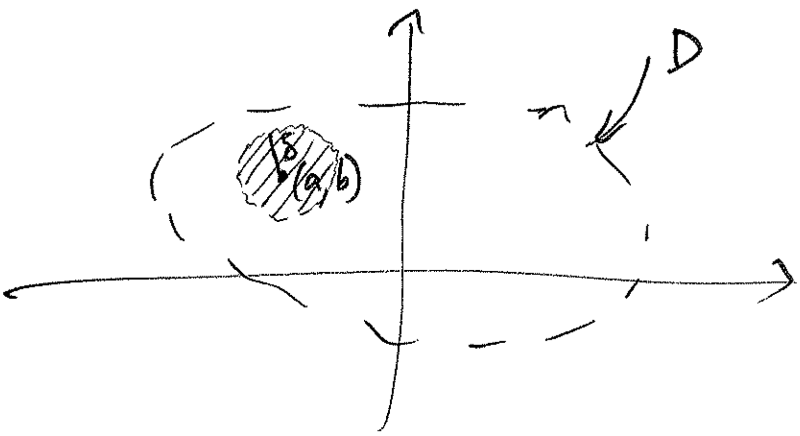
we have

$$|f(x,y) - \ell| < \varepsilon .$$

i.e. $\sqrt{(x-a)^2+(y-b)^2}$

Note

If $|(x,y)-(a,b)| < \delta$ then (x,y) is in disc radius δ around (a,b).



A key difference from ~~funct~~ limits (4)
of functions of one variable is the following.

If $g : I \to \mathbb{R}$ with $I \subseteq \mathbb{R}$ i.e.
g is function of one variable

then $\lim_{x \to a} g(x) = \ell$

if and only if

$\lim_{x \to a^-} g(x) = \ell$ and $\lim_{x \to a^+} g(x) = \ell$.

There are only two ways that x can approach a: from above and from below.

BUT in $\mathbb{R}^2$ there are infinitely many ways that (x,y) can approach the point (a,b). In order for

$\lim_{(x,y) \to (a,b)} f(x,y)$ to exist and equal ℓ

we must have $f(x,y) \to l$ as (x,y) approaches (a,b) along <u>every</u> path.

<u>Example</u>

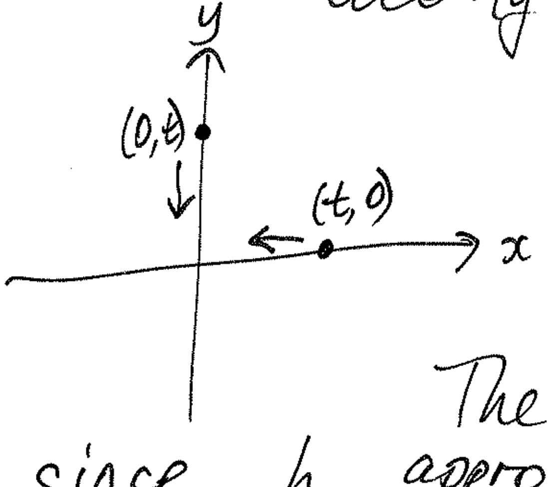
$$h(x,y) = \frac{x^2-y^2}{x^2+y^2}$$

Show that $\lim\limits_{(x,y)\to(0,0)} h(x,y)$ does <u>not</u> exist.

In polar coords $h(x,y) = \cos 2\theta$

Level curves tell us that the limit does not exist.

To show limit does not exist, enough to find 2 ~~paths~~ curves approaching $(0,0)$ so that $h(x,y)$ ~~takes~~ approaches different values along these curves.



For $t>0$, $h(t,0) = \frac{t^2}{t^2} = 1$

$$h(0,t) = \frac{-t^2}{t^2} = -1$$

The limit does not exist since h approaches different values along these paths.

Many things do work the same way for limits of functions of 2 variables.

The following limit laws hold:

~~1. Sum Addition Law~~

If $\lim_{(x,y)\to(a,b)} f(x,y) = l$ and $\lim_{(x,y)\to(a,b)} g(x,y) = m$

then

1. Addition Law

$$\lim_{(x,y)\to(a,b)} [f(x,y) + g(x,y)] = l + m$$

2. Subtraction Law

3. Product Law

4. Quotient Law $\lim_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{l}{m}$

provided $m \neq 0$

~~5. Substitution Law~~

~~If $\lim_{x\to}$~~

5. Squeeze Law: if

$$f(x,y) \leq g(x,y) \leq h(x,y)$$

near (a,b) and as $(x,y) \to (a,b)$

we have $f(x,y) \to l$ and $h(x,y) \to l$ then $g(x,y) \to l$ as $(x,y) \to (a,b)$.

Example

Let $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$.

Domain is $\{(x,y) \mid (x,y) \neq (0,0)\}$.

Show that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$

$$f(x,y) = \frac{r^2 \cos\theta \sin\theta}{\sqrt{r^2(\cos^2\theta + \sin^2\theta)}}$$

$$= r\cos\theta\sin\theta \qquad \text{since } r > 0$$

$$= \frac{1}{2} r \sin 2\theta.$$

Since $-1 \leq \sin 2\theta \leq 1$ we have

$$-\frac{1}{2}r \leq f(x,y) \leq \frac{1}{2}r$$

Now as (x,y) approaches the origin along <u>any</u> path, r approaches 0.

Thus by the Squeeze Law

$$0 = \lim_{r\to 0} \tfrac{1}{2} r \le \lim_{(x,y)\to(0,0)} f(x,y) \le \lim_{r\to 0} \tfrac{1}{2} r = 0$$

so $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0.$

Continuity

Definition If $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$ and $(a,b) \in D$, then f is continuous at the point (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

and f is continuous on D if f is continuous at each point in D.

Example Let

$$g(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \ne (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that g is continuous at $(0,0)$.

We already showed

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$$

thus $\lim_{(x,y)\to(0,0)} g(x,y) = 0 = g(0,0)$ so

g is continuous at $(0,0)$.

Example Is there a real number c so that the function

$$g(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ c & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$?

There is such a c if and only if

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$ exists (and equals c).

Using polar coords

$$\frac{xy}{x^2+y^2} = \sin 2\theta$$

and limit as $(x,y)\to(0,0)$ does not exist.

Examples of continuous functions

1. A monomial in two variables is a function $f: \mathbb{R}^2 \to \mathbb{R}$ of the form
$$f(x,y) = cx^m y^n$$
where $c \in \mathbb{R}$, $m, n \geq 0$ integers
e.g. $-2xy$, $17x^5y^{273}$, x^5

Monomials are continuous.

2. A polynomial in two variables is a $f: \mathbb{R}^2 \to \mathbb{R}$ which is a sum of monomials.

e.g.
$$f(x,y) = x^2 + y^2$$
$$f(x,y) = x^2 + 2xy + y^2$$
$$f(x,y) = 3xy^3 + 3x^3y$$

Polynomials are continuous.

3. Rational functions $\dfrac{f(x,y)}{g(x,y)}$

where f, g polynomials are

continuous on their domains

$$\{(x,y) \mid g(x,y) \neq 0\}$$

e.g. $h(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ is continuous on its domain i.e. continuous at all $(x,y) \neq (0,0)$.

4. Compositions: if $h: \mathbb{R}^2 \to \mathbb{R}$ is continuous and $g: \mathbb{R} \to \mathbb{R}$ then

$$f = g \circ h \quad \text{(composition)}$$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, bend left, "h"] \arrow[rr, bend right, "f"'] & \mathbb{R} \arrow[r, bend left, "g"] & \mathbb{R} \end{tikzcd}$$

is continuous on its domain.

e.g. $h(x,y) = \sqrt{x+y} - 1$

$g(x) = \ln(x)$

then $f(x,y) = \ln(\sqrt{x+y} - 1)$

is continuous on its domain.