Week 10 Lecture 2

Quiz 2 : in week 12, on material in lectures weeks 6-10, tutorials weeks 7-11.

Partial derivatives

Suppose f is a function of ~~3~~ two variables x and y.

Keep y fixed i.e. put $y=b$ a constant.

Define a function of one variable by

$$g(x) = f(x,b).$$

If g has a derivative at $x=a$ then by definition

$$g'(a) = \lim_{h\to 0} \frac{g(a+h)-g(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

Definition The partial derivative of f with respect to x at the point (a,b) is

$$f_x(a,b)$$

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

Now keep x fixed, $x = a$ constant.
Define a function of one variable

$$h(y) = f(a,y).$$

If h is differentiable at $y = b$ then

$$h'(b) = \lim_{h \to 0} \frac{h(b+h) - h(b)}{h}$$

$$= \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}.$$

Def^n The partial derivative of f with respect to y at the point (a,b) is

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}.$$

The partial derivatives of f are the functions f_x and f_y given by

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

To compute f_x and f_y:

- to compute f_x, think of y as a constant and differentiate with respect to x
- to compute f_y, think of x as a constant and differentiate with respect to y

<u>Example</u> Find the partial derivatives of

1. $f(x,y) = 3xy^2 - 2x^5y$

$$f_x(x,y) = 3y^2 - 2y(5x^4)$$
$$= 3y^2 - 10x^4y.$$
$$f_y(x,y) = 3x(2y) - 2x^5(1)$$
$$= 6xy - 2x^5.$$

At the point $(x,y) = (1,1)$

$$f_x(1,1) = 3 - 10 = -7$$
$$f_y(1,1) = 6 - 2 = 4.$$

2. $f(x,y) = x\cos(xy)$

$$f_x(x,y) = (1)\cos(xy) + x\left(-y \overset{\sin}{\cancel{\cos}}(xy)\right)$$

$$f_x(x,y) = \cos(xy) - xy\sin(xy)$$

$$f_y(x,y) = -x^2\sin(xy)$$

(no product rule needed for f_y, product rule needed for f_x)

Alternative notation

if $z = f(x,y)$

$$f_x = f_x(x,y) = \underbrace{\frac{\partial z}{\partial x}}_{\substack{\text{pronounced dee z} \\ \text{by dee x}}} = \frac{\partial f}{\partial x} = \underbrace{\frac{\partial}{\partial x} f(x,y)}_{\substack{\text{dee by} \\ \text{dee x} \\ \text{of } f(x,y)}}$$

$$f_y = f_y(x,y) = \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y).$$

Interpretations of the partial derivative

~~#~~ 1. Rates of change

If $z = f(x,y)$ then $\frac{\partial z}{\partial x}$ is the rate of change of z with respect to x, when y is fixed. So this tells you how the output depends on each ~~index~~ of the inputs by themselves.

Similarly $\cancel{\frac{\partial z}{\partial x}}$ $\frac{\partial z}{\partial y}$ is the rate of change of $\cancel{y}$ z with respect to y when x is fixed.

2. Geometric

Suppose $f(a,b) = c$, then the point $P = (a,b,c)$ lies on the surface

$$z = f(x,y).$$

The set of points on the surface with $y=b$ is the intersection of the surface with the plane $y=b$.

i.e. slice surface along plane $y=b$.

Let C_1 be the curve obtained by ~~the~~ intersecting surface with $y=b$.

Then if $g(x) = f(x,b)$,

$$g'(a) = \frac{\partial}{\partial x} f(a,b)$$ is the slope of the tangent line to C_1 at the ~~point P~~

point $P=(a,b,c)$.

Let C_2 be the curve obtained by intersecting the surface with the plane $x=a$. Then if

$$h(y) = f(a,y)$$

we have that ~~$h'(y) = f$~~

$$h'(b) = f_y(a,b) = \frac{\partial}{\partial y} f(a,b)$$

is the slope of the tangent line T_2 to the curve C_2 at the point $P=(a,b,c)$.

<u>Tangent Plane to a surface at a point</u>

Let S be the surface $z=f(x,y)$. Assume f_x and f_y exist and are continuous at the point $P=(a,b,c)$ (with $c=f(a,b)$) on this surface.

~~Let~~

Let T_1 be tangent line to the curve C_1 which is intersection of surface with plane $y=b$.
Let T_2 be tangent line to the curve C_2 which is intersection of surface with plane $x=a$.

~~Def^n~~ Both T_1 and T_2 are lines through the point P. (and not the same line).

<u>Def^n</u> The <u>tangent plane to S at the point P</u> is the unique plane containing the lines T_1 and T_2.

<u>Fact</u> If C is <u>any</u> curve on the surface S which passes through P then the tangent line to the curve C at P lies in the tangent plane at P.

Finding the equation of the tangent plane at $P = (a,b,c)$: 8

the tangent plane will have equation

$$z - c = k_1(x-a) + k_2(y-b)$$

for some constants k_1, k_2, since it contains the point (a,b,c).

We need to find k_1 and k_2.

Put $y = b$ then

$$(1) \quad z - c = k_1(x-a)$$

is the intersection of the plane $y=b$ with the tangent plane at P. This is a line. It is the tangent line T_1 to the curve C_1.

So it has slope

$$k_1 = \frac{\partial z}{\partial x} = f_x(a,b).$$

Similarly $k_2 = f_y(a,b)$.

Thus the equation of the tangent plane to the surface $z = f(x,y)$ at the point (a,b,c) is

$$z - c = f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

<u>Example</u> Let $f(x,y) = 2x^2 + y^2$.

$$f_x(x,y) = 4x$$

$$f_y(x,y) = 2y$$

~~Then~~ Find the eq^n of the tangent plane to this surface at $(1,1,3)$

$$f_x(1,1) = 4 \qquad f_y(1,1) = 2$$

Eq^n of tangent plane is

$$z - 3 = 4(x-1) + 2(y-1).$$

Quiz won't include higher partial derivatives $f_{xx}, f_{xy}, f_{yx}, f_{yy}$.