# Week 11 Lecture 1

Recall the partial derivatives $z = f(x,y)$

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

These are functions of $x$ and $y$, so we can find their partial derivatives.

4 possibilities:

1. differentiate twice w.r.t. $x$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial^2 z}{\partial x^2} = f_{xx}(x,y) = f_{xx}$$

2. differentiate twice w.r.t. $y$

$$\cancel{\frac{\partial}{\partial x}\left(\partial\right.} \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x,y)$$

$$= f_{yy}$$

3. differentiate first w.r.t. $x$ then w.r.t. $y$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\,\partial x} = (f_x)_y = f_{xy}$$ Yuck

4. Differentiate first w.r.t. $y$ then w.r.t. $x$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\,\partial y} = (f_y)_x = f_{yx}.$$

<u>Example</u>

Find all second partial derivatives of

$$f(x,y) = 3xy^2 - 2x^5 y$$

$$\frac{\partial f}{\partial x} = 3y^2 - 10x^4 y \qquad \frac{\partial f}{\partial y} = 6xy - 2x^5$$

$$\frac{\partial^2 f}{\partial x^2} = -40x^3 y \qquad \frac{\partial^2 f}{\partial y^2} = 6x$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = 6y - 10x^4 \qquad \frac{\partial^2 f}{\partial x\,\partial y} = 6y - 10x^4$$

same

mixed partial derivatives

**Theorem** If the mixed partial derivatives

$$\frac{\partial^2 f}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x}$$

are both continuous at each point in some disc centred at $(a,b)$ then they are the same at the point $(a,b)$

i.e. $$\frac{\partial^2 f}{\partial x \partial y}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b).$$

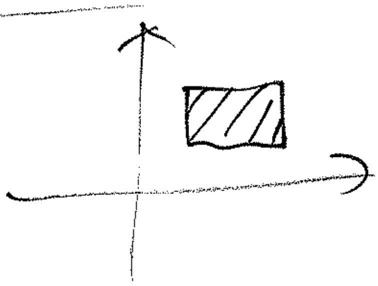
## Finding maxima and minima of functions of two variables

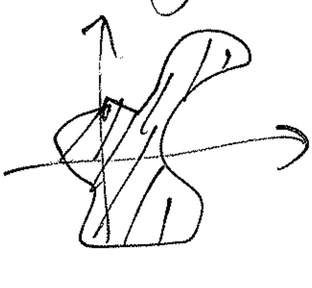
**Extreme Value Theorem**

Let $D$ be the set of all points on and inside a simple closed curve in the xy-plane.

loop which doesn't intersect itself

Formally, if $f$ is an injective function from the circle to the plane, image of





A s.c.c. (simple closed curve) is the image of an injective function from the circle to the xy-plane.



If $f: D \to \mathbb{R}$ is continuous on $D$ then $f$ attains its absolute max. and absolute min. on $D$.

i.e. there is a point $(a,b) \in D$ so that for all $(x,y) \in D$

$$f(a,b) \geq f(x,y)$$ (absolute max.)

and there is a point $(c,d) \in D$ so that for all $(x,y) \in D$

$$f(c,d) \leq f(x,y)$$ (absolute min.).

<u>Theorem</u> Suppose $f$ has a local max. or a local min. at $(a,b)$ in the interior of $D$.

If the partial derivatives

$f_x(a,b)$ and $f_y(a,b)$

both exist, then

$f_x(a,b)=0$ and $f_y(a,b)=0$.

Notes

1. The absolute max. or min. could occur on the boundary of D.
2. If $f_x(a,b)=0$ and $f_y(a,b)=0$ there might not be a local max. or a local min. at $(a,b)$. Analogue of a point of inflection is called a saddle point.

~~Example~~ 3. Recall the equation ~~Fm~~ of the tangent plane to the $z=f(x,y)$ at the point $(a,b,f(a,b))$ is is

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

This tangent plane is horizontal if and only if both $f_x(a,b)=0$ and $f_y(a,b)=0$.

In particular, at a local max or local min, or at a saddle point, the tangent plane is horizontal.

<u>Example</u> Find any local max. or min. of

$$f(x,y) = 3x^2 + 4xy + 4y^2 + 2x + 12y + 5.$$

$$f_x(x,y) = 6x + 4y + 2 \quad f_y(x,y) = 4x + 8y + 12$$

The points with $f_x(x, y) = 0$ and $f_y(x,y) = 0$ are the solutions to the simultaneous equations

$$\begin{cases} 6x + 4y + 2 = 0 \\ 4x + 8y + 12 = 0 \end{cases}$$

Here the only solution is $(x,y) = (1,-2)$.
Thus at the point $(1,-2,-6)$ on the surface $z = f(x,y)$, we have both partial derivatives equal to $0$.
Is this a local max, a local min, or neither?
If for all $h, k$ close to $0$, ~~and~~ we have

$$f(1+h, -2+k) \geq f(1,-2)$$

then we have local minimum at $(1,-2)$

If for all $h, k$ close to $0$ 8

we have

$$f(1+h, -2+k) \leq f(1,-2)$$

then we have a local max. at $(1,-2)$.

If neither, saddle point.

$$f(1+h, -2+k) = 3h^2 + 4hk + 4k^2 - 6$$

after simplification

So $f(1+h, -2+k) - f(1,-2)$

$$= 3h^2 + 4hk + 4k^2 - 6 + 6$$

$\vdots$ complete the square

$\geq 0$ since a sum of squares

hence local min at $(1,-2)$.