Week 11 Lecture 2

- Partial differentiation vs implicit differentiation

Partial differentiation:

$\frac{\partial}{\partial x}$ means treat y as constant, differentiate w.r.t. x

e.g if $x^2+y^2=1$ then

$$2x=0$$

is obtained by partial differentiation

Implicit differentiation:

treat y as a function of x and differentiate w.r.t. x

e.g. if $x^2+y^2=1$

then $2x+2y\frac{dy}{dx}=0$

~~is the derivat~~

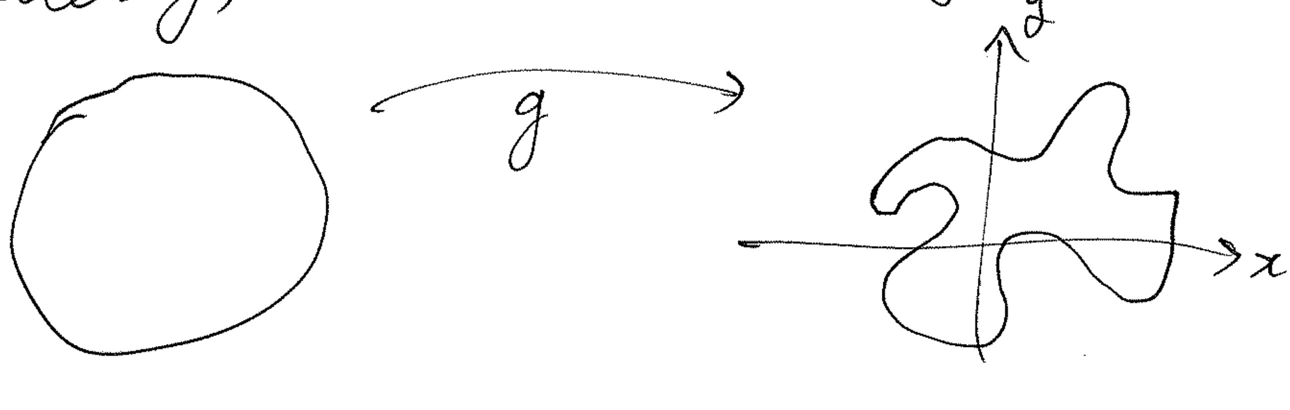
is obtained by implicit differentiation

- Why is a non-intersecting loop in the xy-plane the same thing as an injective function

$$g: \text{circle} \longrightarrow xy\text{-plane} \quad ?$$

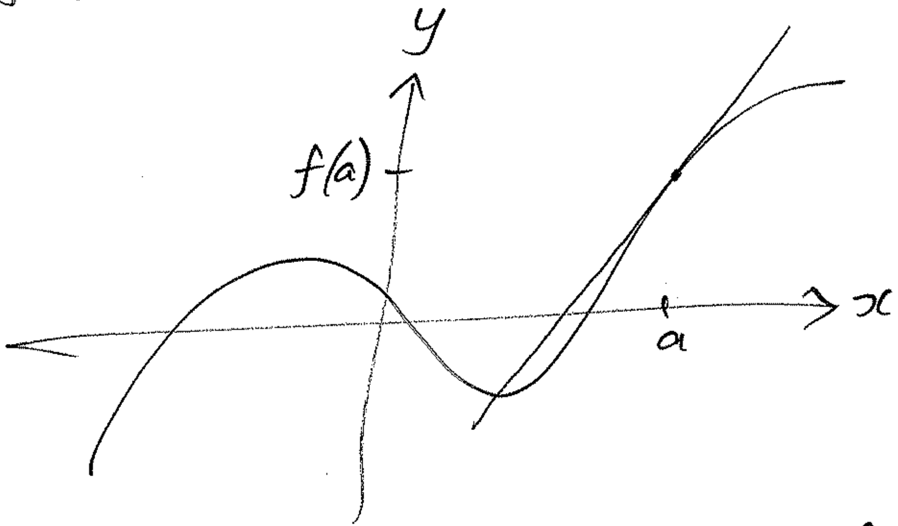
Recall: g is injective if wherever $g(x_1)=g(x_2)$ we have $x_1=x_2$.

Equivalently, if $x_1 \neq x_2$ then $g(x_1) \neq g(x_2)$.



Linear Approximations

If f is a function of one variable and f is differentiable at a, then the tangent line at $(a, f(a))$ is the best straight line approximation to the graph of f near $(a, f(a))$.



If $f(x,y)$ is a function of two variables, then the tangent plane at $(a, b, f(a,b))$ is the best planar approximation to the graph of f near $(a, b, f(a,b))$.

If $P = (a, b, f(a,b))$ then the tangent plane at P has equation

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

So for points Q which are on the surface $z = f(x,y)$ and are <u>near</u> P, with $Q = (z, y, f(x,y))$, the z-value at Q is approximately

$$z \overset{\text{approximately}}{\approx} f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

i.e. we're using the z-value on the tangent plane at P, which will be above or below Q, to approximate the z-value at Q.

The <u>linear approximation</u> to $f(x,y)$ near (a,b) is

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

<u>Example</u> Use the linear approximation of a suitable function to estimate

$$0.98\sqrt{4.03}\,.$$

The function to use is

$$f(x,y) = x\sqrt{y} \qquad (\text{or } y\sqrt{x})$$

then

$$f(0.98, 4.03) = 0.98\sqrt{4.03}$$
$$\approx f(1,4) = 2$$

Since $(0.98, 4.03)$ is close to $(1,4)$

the linear approximation is

$$f(0.98, 4.03) \approx f(1,4) + f_x(1,4)(0.98-1)$$
$$+ f_y(1,4)(4.03-4).$$

$$f_x(x,y) = \sqrt{y} \qquad f_y(x,y) = \frac{x}{2\sqrt{y}}$$

$$f_x(1,4) = 2 \qquad f_y(1,4) = \frac{1}{4}$$

So

$$f(0.98, 4.03) \approx 2 + 2(0.98-1) + \frac{1}{4}(4.03-4)$$
$$= 1.9675.$$

Partial Differential Equations (PDEs)

PDEs are equations involving a function and its partial derivatives.

Example Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Solutions to this equation are called harmonic functions.

Note ~~Sums of ha~~

If u_1 and u_2 are harmonic, then

$u_1 + u_2$ is harmonic.

If u is harmonic and c is a scalar then cu is harmonic.

Example Show that $u(x,y) = e^x \sin y$ is a harmonic function.

$$u(x,y) = e^x \sin y$$

$$\frac{\partial u}{\partial x} = e^x \sin y \qquad \frac{\partial u}{\partial y} = e^x \cos y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \sin y \qquad \frac{\partial^2 u}{\partial y^2} = -e^x \sin y$$

So $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \sin y - e^x \sin y$$
$$= 0$$

thus u is a solution to Laplace's equation. So u is harmonic.

Side note: if $z = x + iy$

$$e^z = e^{x+iy}$$
$$= e^x e^{iy}$$
$$= e^x(\cos y + i \sin y)$$

$\mathrm{Re}(e^z) = e^x \cos y$ and $\mathrm{Im}(e^z) = e^x \sin y$

both are harmonic

Chain Rule for functions of two variables

For functions of one variable:

suppose

$$y = f(u) \qquad u = g(x)$$

are differentiable, then

$$y = (f \circ g)(x) = f(g(x))$$

is differentiable and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$y' = f'(g(x))\,g'(x).$$

<u>1st version</u> for functions of 2 variables.

Suppose x and y are functions of a single variable t, $x = g(t)$, $y = h(t)$

e.g. $x = \cos t \qquad y = \sin t$.

Suppose z is a function of two

variables x and y

$$z = f(x,y).$$

Assume ~~xx~~ g and h are differentiable and that f_x and f_y exist and are continuous.

Then z is a differentiable function of the single variable t

$$z = f(x,y) = f(g(t), h(t))$$

and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= f_x \frac{dx}{dt} + f_y \frac{dy}{dt}.$$

<u>Example</u> Let $z = xy$ where

$x = \cos t$ $\qquad$ $y = \sin t.$

Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$ (9)

$$= (y)(-\sin t) + (x)(\cos t)$$

$$= -\sin^2 t + \cos^2 t$$

i.e. put everything in terms of t.

Alternatively:

$$z = xy = \cos t \sin t$$

then by product rule

$$\frac{dz}{dt} = (-\sin t)(\sin t) + (\cos t)(\cos t)$$

$$= \cos^2 t - \sin^2 t.$$

<u>2nd version</u> of chain rule for functions of 2 variables.

Suppose $x = g(s,t)$ and $y = h(s,t)$

e.g. $x = r\cos\theta \qquad y = r\sin\theta$

variables are r, θ

are functions of 2 variables.

Suppose $z = f(x,y)$ is a function of 2 variables x and y.

Then $z = f(x,y) = f(g(s,t), h(s,t))$

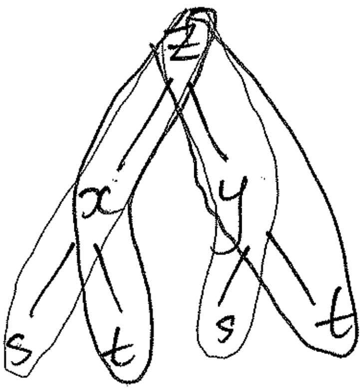
is a function of s and t.

Assume all partial derivatives exist and are continuous.

Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$



<u>Example</u> $z = xy - x^2e^y$

and $x = r\cos\theta$, $y = r\sin\theta$

Then

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= (y - 2xe^y)(-r\sin\theta)$$
$$\quad + (x - x^2 e^y)(r\cos\theta)$$
$$= (r\sin\theta - 2r\cos\theta\, e^{r\sin\theta})(-r\sin\theta)$$
$$\quad + (r\cos\theta - r^2\cos^2\theta\, e^{r\sin\theta})(r\cos\theta)$$
$$= r^2\cos 2\theta + r^2 e^{r\sin\theta}(\sin 2\theta - r\cos^3\theta).$$

Alternatively:

$$z = xy - x^2 e^y$$
$$= (r\cos\theta)(r\sin\theta) - (r\cos\theta)^2 e^{r\sin\theta}$$

and compute partial derivatives from here.