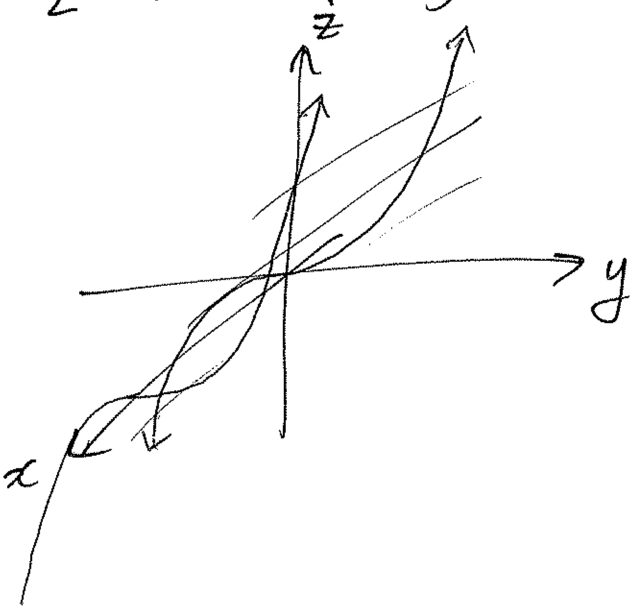
Week 12 Lecture 1

If $f_x(a,b)=0$ and $f_y(a,b)=0$ then at point $(a, b, f(a,b))$ on the surface $z=f(x,y)$, could have

- local max
- local min
- saddle point 
- something else

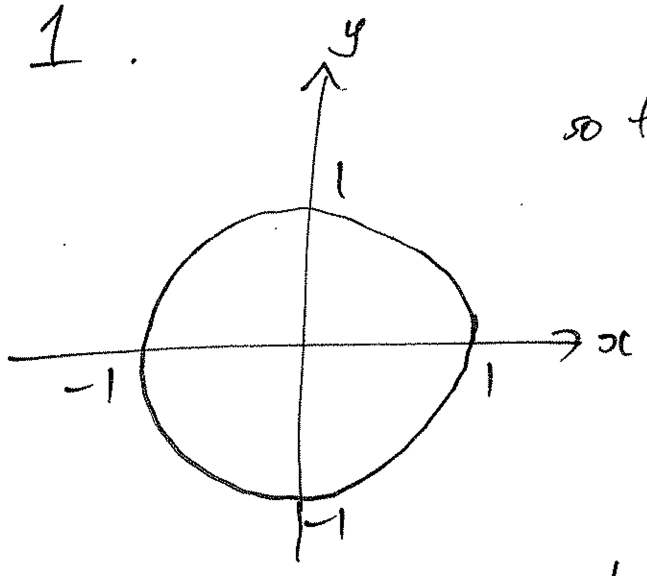
e.g. $z = x^3 = f(x,y)$



Implicit Differentiation and the Implicit Function Theorem

Important example

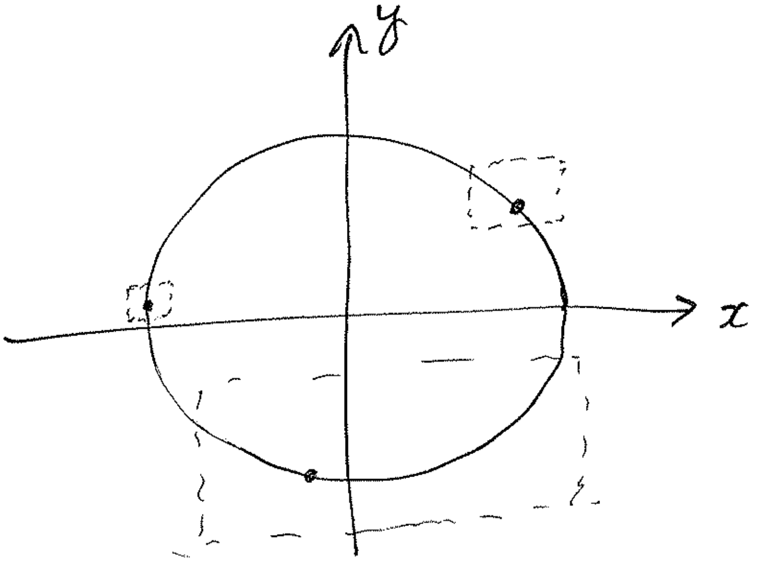
Let $f(x,y) = x^2+y^2$. Consider the level curve of the surface $z=f(x,y)$ at height 1.



Level curve is (x,y) so that $x^2+y^2=1$.

This curve in the xy-plane is <u>not</u> the graph of any real-valued function of one variable $y=g(x)$, since it fails the vertical line test.

However for all points (a,b) on this curve with $b \neq 0$, we can find a box around (a,b) so that inside the box, the curve <u>is</u> the graph of a function.



Note: no such box around $(1,0)$ or $(-1,0)$.

In this particular example, but not in general, we can find the function explicitly.

If $b > 0$, use $y = \sqrt{1-x^2}$

If $b < 0$, use $y = -\sqrt{1-x^2}$

We'd like to find slope of tangent line to this curve at (a,b), without necessarily finding an explicit formula for the function near (a,b). i.e. find $\frac{dy}{dx}$, without knowing formula for y as a function of x.

To do this, use partial derivatives.

Curve is given by equation

$$f(x,y) = 1.$$

Think of both x and y as functions of a single variable x, and use Chain Rule for $\frac{\partial}{\partial x}$.

$$\frac{\partial}{\partial x}\left(f(x,y)\right) = \frac{\partial}{\partial x}(1)$$

$$\frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

$$f_x(x,y).1 + f_y(x,y)\underbrace{\frac{dy}{dx}}_{\text{slope of tangent line}} = 0$$

Assume $f_y(x,y) \neq 0$ then

$$\frac{dy}{dx} = \frac{-f_x(x,y)}{f_y(x,y)}.$$

Hence if $f_y(a,b) \neq 0$ then the slope of the tangent line to the curve $f(x,y)=1$ at the point (a,b) is

$$\frac{dy}{dx} = \frac{-f_x(a,b)}{f_y(a,b)}.$$

In the example $f(x,y) = x^2+y^2 = 1$

$$f_x(x,y) = 2x \qquad f_y(x,y) = 2y$$

Note $f_y(x,y) \neq 0 \iff y \neq 0$.

That is, points (a,b) where curve $f(x,y) = 1$ is <u>not</u> locally the graph of a function are exactly the points with $f_y(a,b) = 0$.

So for example the slope of the tangent line to $x^2+y^2=1$ at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is

$$\frac{dy}{dx} = \frac{-2\left(\frac{1}{2}\right)}{2\left(\frac{\sqrt{3}}{2}\right)} = -\frac{1}{\sqrt{3}}.$$

Implicit Function Theorem

Assume f_x and f_y exist and are continuous.

Let (a,b) be a point on the level curve at height k of the surface $z = f(x,y)$.

If $f_y(a,b) \neq 0$ then there is a box in the xy-plane containing (a,b) so that

- within this box the level curve is the graph of a differentiable function $y = g(x)$, and
- $$\left.\frac{dy}{dx}\right|_{(a,b)} = g'(x) = \frac{-f_x(a,b)}{f_y(a,b)}.$$

proved in Example.

Example

Let $f(x,y) = x\cos y + y\cos x$.

Find the equation of the tangent line to the level curve of the surface $z = f(x,y)$ at height 1, at the point $(0,1)$.

The level curve at height 1 is

$$x\cos y + y\cos x = 1.$$

(impossible to write formula for y).

$$f_x(x,y) = \cos y - y\sin x$$

$$f_y(x,y) = -x\sin y + \cos x$$

At the point $(0,1)$

$$f_x(0,1) = \cos 1$$

$f_y(0,1) = 1 \neq 0$ so we may apply the Implicit Function Thm.

At $(0,1)$

$$\frac{dy}{dx} = \frac{-f_x(0,1)}{f_y(0,1)}$$

$$= -\cos 1 .$$

Equation of tangent line at $(0,1)$ is

$$y - 1 = (-\cos 1)(x - 0)$$

$$y = (-\cos 1)x + 1 .$$

Another example

Find $\frac{dy}{dx}$ where y is defined implicitly as a function of x by

$$(*) \quad 2y^2 + \sqrt[3]{xy} = 3x^2 + 17 .$$

Let $f(x,y) = 2y^2 + \sqrt[3]{xy} - 3x^2$.

Then the equation $(*)$ can be viewed as the level curve at height 17 of the surface $z = f(x,y)$.

$$f_x = \frac{1}{3}y(xy)^{-2/3} - 6x , \quad f_y = 4y + \frac{1}{3}x(xy)^{-2/3}$$

So $$\frac{dy}{dx} = \frac{-f_x}{f_y}$$

$$= \frac{-\dfrac{y}{3(xy)^{2/3}} + 6x}{4y + \dfrac{x}{3(xy)^{2/3}}}.$$

Directional derivatives

Consider the surface $z = f(x,y)$.
Assume f_x and f_y exist and are continuous. Recall the def^n's

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

and if $P = (a,b,c)$ with $c = f(a,b)$, then

- $f_x(a,b)$ is slope of tangent line at P to the curve C_1 obtained by intersecting surface with plane $y=b$.

- $f_y(a,b)$ is slope of the tangent line to the curve C_2 that we got by intersecting surface with the plane $x=a$.

If we instead consider the one-sided limit

$$\lim_{h\to 0^+} \frac{f(a+h,b)-f(a,b)}{h}$$

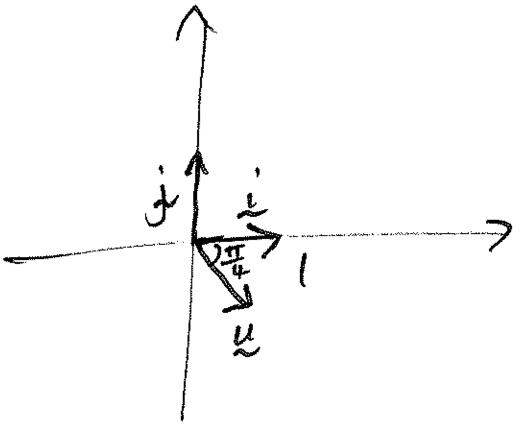
this is the rate of change of z (height) if we walk in the direction of the vector $\underset{\sim}{i}=(1,0)$, which points along the positive x-axis.

What about if we walk in other directions?

We will describe directions using unit vectors i.e. vectors of length 1 i.e. $\underset{\sim}{u}=u_1\underset{\sim}{i}+u_2\underset{\sim}{j}$ where $u_1^2+u_2^2=1$.

e.g. $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are unit vectors

$\underset{\sim}{u} = \frac{1}{\sqrt{2}} \underset{\sim}{i} - \frac{1}{\sqrt{2}} \underset{\sim}{j}$ is a unit vector



<u>Definition</u> The <u>directional derivative</u> of $f(x,y)$ at the point (a,b), in the direction of the unit vector

$\underset{\sim}{u} = u_1 \underset{\sim}{i} + u_2 \underset{\sim}{j}$ is

$$D_{\underset{\sim}{u}} f(a,b) = \lim_{h \to 0^+} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

<u>Note</u> 1. $(a+hu_1, b+hu_2) = (a,b) + h(u_1, u_2)$
$= (a,b) + h\underset{\sim}{u}$

2. the one-sided partial derivatives are a special case

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eg. if $\underset{\sim}{u} = \underset{\sim}{i} = 1\underset{\sim}{i} + 0\underset{\sim}{j}$

$$D_{\underset{\sim}{i}} f(a,b) = \lim_{h \to 0^+} \frac{f(a+h, b) - f(a,b)}{h}.$$