Week 12 Lecture 2

Definition The directional derivative of $f(x,y)$ at the point (a,b) in the direction of the unit vector $\underset{\sim}{u} = u_1 \underset{\sim}{i} + u_2 \underset{\sim}{j}$ is

$$D_{\underset{\sim}{u}} f(a,b) = \lim_{h \to 0^+} \frac{f(a + hu_1, b + hu_2) - f(a,b)}{h}$$

Example Let $f(x,y) = xy$.

Find $D_{\underset{\sim}{u}} f(a,b)$.

$$D_{\underset{\sim}{u}} f(a,b) = \lim_{h \to 0^+} \frac{(a+hu_1)(b+hu_2) - ab}{h}$$

$$= \lim_{h \to 0^+} \frac{ab + h(au_2 + bu_1) + h^2 u_1 u_2 - ab}{h}$$

$$= \lim_{h \to 0^+} \left[(au_2 + bu_1) + hu_1 u_2 \right]$$

$$= au_2 + bu_1.$$

If you are standing on the surface $z = xy$ at the point $(1,3,3)$, and you walk in the direction (2)

given by the unit vector $\underset{\sim}{u} = \frac{1}{\sqrt{2}}\underset{\sim}{i} + \frac{1}{\sqrt{2}}\underset{\sim}{j}$,

will you go up or down?

If $D_{\underset{\sim}{u}}f(1,3) > 0$, go up.

If $D_{\underset{\sim}{u}}f(1,3) < 0$, go down.

If $D_{\underset{\sim}{u}}f(1,3) = 0$, stay level.

$$D_{\underset{\sim}{u}}f(\overset{a}{1},\overset{b}{3}) = \frac{1}{\sqrt{2}} + \frac{3}{\sqrt{2}} > 0$$ so you go up.

$u_1 = \frac{1}{\sqrt{2}}$ $u_2 = \frac{1}{\sqrt{2}}$

The gradient vector function

The gradient of $f(x,y)$ is the vector-valued function

$$\underset{\sim}{\nabla} f(x,y) = f_x(x,y)\,\underset{\sim}{i} + f_y(x,y)\,\underset{\sim}{j}.$$

Pronounced "grad f".

We can use $\underset{\sim}{\nabla} f$ to compute the directional derivative. (3)

<u>Theorem</u> $\quad D_{\underset{\sim}{u}} f(x,y) = \underset{\sim}{\nabla} f(x,y) \cdot \underset{\sim}{u}$

dot product

$\underset{\sim}{u} = u_1 \underset{\sim}{i} + u_2 \underset{\sim}{j}$ unit vector

$$= f_x(x,y) u_1 + f_y(x,y) u_2.$$

<u>Proof</u>

Define $g(h) = f(x + u_1 h, y + u_2 h)$.

Then g is a function of a single variable h, so

$$g'(0) = \lim_{k \to 0^+} \frac{g(k) - g(0)}{k}$$

$$= \lim_{k \to 0^+} \frac{f(x + u_1 k, y + u_2 k) - f(x,y)}{k}$$

$$= D_{\underset{\sim}{u}} f(x,y).$$

Now let $A(h) = x + u_1 h$, $B(h) = y + u_2 h$.

Then A, B can be viewed as functions of h so $g(h) = f(A(h), B(h))$ can be differentiated using the Chain Rule

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh}$$

$$= f_x(x,y)\,u_1 + f_y(x,y)\,u_2$$

So $g'(0) = f_x(x,y)u_1 + f_y(x,y)u_2$

Thus $D_{\underset{\sim}{u}} f(x,y) = f_x(x,y)u_1 + f_y(x,y)u_2$

$$= (f_x(x,y), f_y(x,y)) \cdot (u_1, u_2)$$

$$= \underset{\sim}{\nabla} f(x,y) \cdot \underset{\sim}{u}. \quad //$$

<u>Example</u> If $f(x,y) = xy$

then $\underset{\sim}{\nabla} f(x,y) = y\underset{\sim}{i} + x\underset{\sim}{j}$

so $\underset{\sim}{\nabla} f(1,3) = 3\underset{\sim}{i} + \underset{\sim}{j}$

So $D_{\underset{\sim}{u}} f(1,3) = (3\underset{\sim}{i} + \underset{\sim}{j}) \cdot (\frac{1}{\sqrt{2}}\underset{\sim}{i} + \frac{1}{\sqrt{2}}\underset{\sim}{j})$

$\underset{\sim}{u} = \frac{1}{\sqrt{2}}\underset{\sim}{i} + \frac{1}{\sqrt{2}}\underset{\sim}{j}$ $\quad = \frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}}.$

Example

Find the directional derivative of $f(x,y) = \ln(x+y) + xy$ in the direction of $\underset{\sim}{v} = \underset{\sim}{i} + 2\underset{\sim}{j}$ at the point $(1,1)$.

Since $\underset{\sim}{v}$ is not a unit vector, we find a unit vector in the same direction as $\underset{\sim}{v}$:

$$\underset{\sim}{u} = \frac{\underset{\sim}{v}}{|\underset{\sim}{v}|} = \frac{\underset{\sim}{i} + 2\underset{\sim}{j}}{\sqrt{5}} = \frac{1}{\sqrt{5}}\underset{\sim}{i} + \frac{2}{\sqrt{5}}\underset{\sim}{j}$$

Then to find $D_{\underset{\sim}{u}} f(1,1)$:

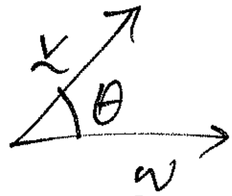
$$\underset{\sim}{\nabla} f(x,y) = \left(\frac{1}{x+y} + y\right)\underset{\sim}{i} + \left(\frac{1}{x+y} + x\right)\underset{\sim}{j}$$

$$\underset{\sim}{\nabla} f(1,1) = \left(\frac{1}{2} + 1\right)\underset{\sim}{i} + \left(\frac{1}{2} + 1\right)\underset{\sim}{j}$$

$$= \frac{3}{2}\underset{\sim}{i} + \frac{3}{2}\underset{\sim}{j}$$

So $D_{\underline{u}} f(1,1) = \left(\frac{3}{2}\underset{\sim}{i} + \frac{3}{2}\underset{\sim}{j}\right) \cdot \left(\frac{1}{\sqrt{5}}\underset{\sim}{i} + \frac{2}{\sqrt{5}}\underset{\sim}{j}\right) = \frac{9}{2\sqrt{5}}$

Greatest and Least Values of $D_{\underset{\sim}{u}} f(x,y)$



Recall $\underset{\sim}{v} \cdot \underset{\sim}{w} = |\underset{\sim}{v}||\underset{\sim}{w}| \cos\theta$

so

$$D_{\underset{\sim}{u}} f(x,y) = \underset{\sim}{\nabla} f(x,y) \cdot \underset{\sim}{u}$$

$$= |\underset{\sim}{\nabla} f(x,y)| \, \overbrace{|\underset{\sim}{u}|}^{=1} \cos\theta$$

where θ is angle between $\underset{\sim}{u}$ and $\underset{\sim}{\nabla} f(x,y)$

$$= |\underset{\sim}{\nabla} f(x,y)| \cos\theta$$

Thus $D_{\underset{\sim}{u}} f(x,y)$ is maximum when $\cos\theta = 1$, and its maximum value is $|\underset{\sim}{\nabla} f(x,y)|$.

Now $\cos\theta = 1 \iff \theta = 0$

$\iff \underset{\sim}{u}$ and $\underset{\sim}{\nabla} f(x,y)$ point in the same direction.

Thus at a point (a,b), the direction of the greatest slope is $\underset{\sim}{\nabla} f(a,b)$, and the value of the greatest slope is ~~$\nabla f(a,y)$~~ $|\underset{\sim}{\nabla} f(a,b)|$ i.e. length of gradient vector at (a,b).

The minimum value of $D_{\underset{\sim}{u}} f(x,y)$ occurs when $\cos\theta = -1$ and this minimum value equals $-|\underset{\sim}{\nabla} f(x,y)|$.

Now $\cos\theta = -1 \iff \theta = \pi$

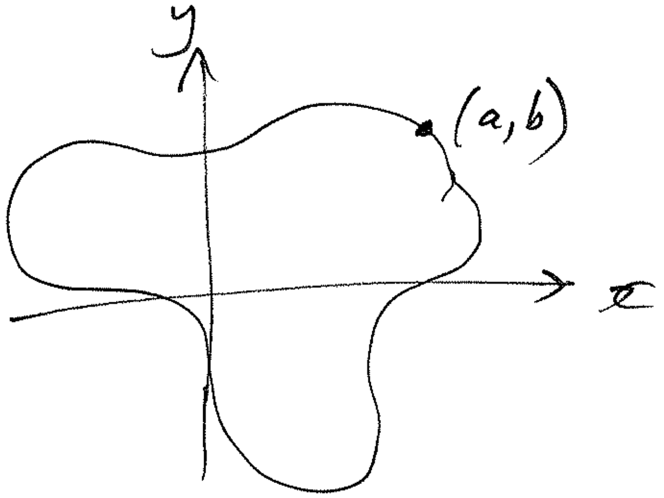
$\iff \underset{\sim}{u}$ and $\underset{\sim}{\nabla} f(x,y)$ point in opposite directions

$\iff \underset{\sim}{u} = -\underset{\sim}{\nabla} f(x,y)$.

So to go down the hill as fast as possible, walk in the direction $-\underset{\sim}{\nabla} f(x,y)$.

<u>The gradient vector and level curves</u>

Consider the level curve $f(x,y) = k$ of the surface $z = f(x,y)$.



<u>Claim</u> At each point (a,b) on the level curve, the vector $\underset{\sim}{\nabla} f(a,b)$ is perpendicular to the curve.

<u>Consequence</u>: if you want to stay level on a hill, at the point (a,b), walk in the two directions perpendicular to

$\underset{\sim}{\nabla} f(a,b)$.

e.g. $\underset{\sim}{\nabla} f(a,b) = 3\underset{\sim}{i} + 2\underset{\sim}{j}$

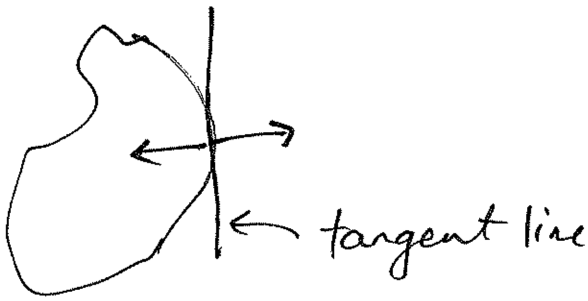
then directions in which you can walk and stay level are

$(-2,3)$ and $(2,-3)$. (their dot product with $\underset{\sim}{\nabla} f(a,b)$ is 0.)

Why is the Claim true?

2 cases:

1. If $f_y(a,b) \neq 0$, use Implicit Function Theorem.
   - this gives slope of tangent line to the level curve at (a,b).
2. If $f_y(a,b) = 0$ then tangent line to level curve is vertical line, while $\underset{\sim}{\nabla} f(x,y) = f_x(x,y)\, \underset{\sim}{i}$ is horizontal.



END OF EXAMINABLE MATERIAL.