Week 2, Lecture 1

$(-\pi, \pi]$

Solutions of polynomial equations

A polynomial over $\mathbb{C}$ is a function

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

where the coefficients $a_0, a_1, a_2, \ldots, a_n \in \mathbb{C}$.

Example

1. constant function $p(z) = a$ where $a \in \mathbb{C}$.
2. identity function $p(z) = z$.
3. $p(z) = z^2 + 3z - 17$
4. $p(z) = z^n$ where n is a non-negative integer.

Finding the n^{th} roots of a complex number

Note: finding solutions of $z^n = w$ is same as finding solutions to the polynomial equation $\underbrace{z^n - w}_{\text{this is a polynomial}} = 0$.

Theorem For any $w \in \mathbb{C}$, and any positive integer n, there are exactly n different complex numbers z such that $z^n = w$.

Proof

Let $w = s \operatorname{cis} \phi$.

Then $z = r \operatorname{cis} \theta$ satisfies $z^n = w$

$\iff$ $r^n \operatorname{cis}(n\theta) = s \operatorname{cis} \phi$ (de Moivre's Theorem)

(if and only if)

$\iff$ $r^n = s$ and $\operatorname{cis}(n\theta) = \operatorname{cis} \phi$

$\iff$ $r = \sqrt[n]{s}$ (positive ~~#~~ n^{th} root) and $n\theta = \phi + 2k\pi$ for some $k \in \mathbb{Z}$

$\iff$ $r = \sqrt[n]{s}$ and $\theta = \dfrac{\phi + 2k\pi}{n}$ for some $k \in \mathbb{Z}$

To count the n^{th} roots of w, we need to see how many of the

$$z_k := \sqrt[n]{s} \operatorname{cis}\left(\frac{\phi + 2k\pi}{n}\right)$$

are distinct, as k varies over $\mathbb{Z}$.

Check: if $k \in \{0, 1, \ldots, n-1\}$ then the ~~values z~~ z_k are all distinct,

so $z^n = w$ has at least n solutions.

To see that $z^n = w$ has at most n solutions:

Write $k = nq + k'$ where $q \in \mathbb{Z}$,

and $k' \in \{0, 1, \ldots, n-1\}$. (remainder term)

Then $\operatorname{cis}\left(\frac{\phi + 2k\pi}{n}\right) = \operatorname{cis}\left(\frac{\phi + 2(nq+k')\pi}{n}\right)$ ③

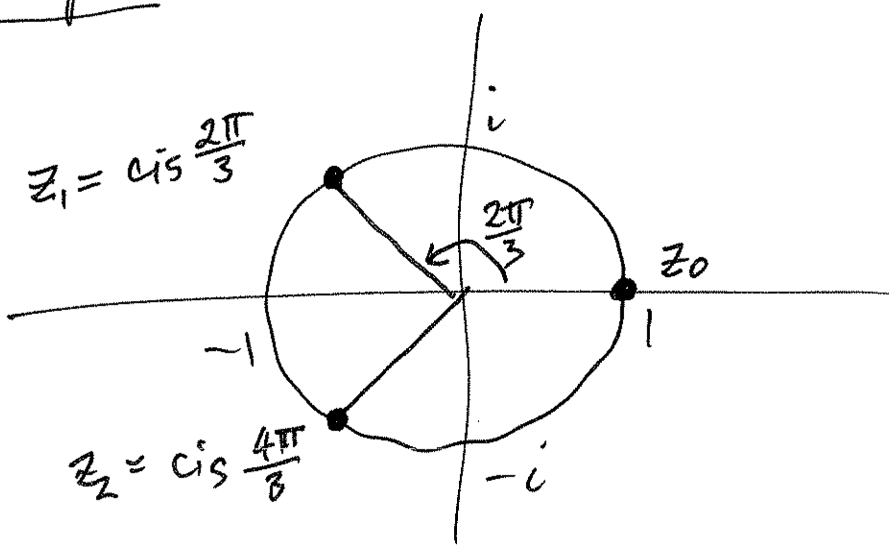
$$= \operatorname{cis}\left(\frac{\phi+2k'\pi}{n} + 2\pi q\right)$$

$$= \operatorname{cis}\left(\frac{\phi + 2k'\pi}{n}\right)$$

ie. for any $k \in \mathbb{Z}$, there is a $k' \in \{0, \dots, n-1\}$ so that $z_k = z_{k'}$. So $z^n = w$ ~~hat~~ has at most n solutions. Therefore ~~ie~~ all solutions of $z^n = w$ are given by $\{z_0, z_1, \dots, z_{n-1}\}$, which has n elements.

Proof shows that solutions to $z^n = w$ lie on circle centre O, radius $|w|^{\frac{1}{n}}$, and are spaced out ~~by~~ evenly with $\frac{2\pi}{n}$ difference in their argument.

<u>Example</u> $z^3 = 1$ has solutions 1, $\operatorname{cis}\frac{2\pi}{3}$, $\operatorname{cis}\frac{4\pi}{3}$.



Fundamental Theorem of Algebra

Let n be a positive integer and $a_0, a_1, \dots, a_{n-1} \in \mathbb{C}$. Then the complex polynomial

$$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_{n-1} z^{n-1} + z^n$$

has at least one root in $\mathbb{C}$

ie. there is at least one $\alpha \in \mathbb{C}$ such that $p(\alpha) = a_0 + a_1\alpha + \dots + a_{n-1}\alpha^{n-1} + \alpha^n = 0$.

Proof see Spivak.

Corollary For p as above, there are complex numbers $\alpha_1, \dots, \alpha_n$, not necessarily all different, so that p factors as

$$p(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)$$

ie. $\alpha_1, \dots, \alpha_n$ are the roots of p.

Example $z^3 - 1 = (z-1)\left(z - \operatorname{cis}\frac{2\pi}{3}\right)\left(z - \operatorname{cis}\frac{4\pi}{3}\right)$

since 1, $\operatorname{cis}\frac{2\pi}{3}$ and $\operatorname{cis}\frac{4\pi}{3}$ are the solutions of $z^3 - 1 = 0$.

Proof of Corollary

Induction on n.

Case $n=1$. We have

$$p(z) = a_0 + z^1 \qquad a_0 \in \mathbb{C}$$
$$= z - (-a_0).$$

For inductive step, by Fund. Thm. Alg. we have $\alpha \in \mathbb{C}$ so that $p(\alpha) = 0$. Now do long division of polynomials:

$$\begin{array}{r|l} & q(z) \\ \hline z-\alpha & p(z) \\ & (z-\alpha)q(z) \\ \hline & p(z)-(z-\alpha)q(z) \end{array}$$

so putting $r(z) = p(z) - (z-\alpha)q(z)$ for the remainder term we have

$$p(z) = (z-\alpha)q(z) + r(z)$$

where $\text{degree}(q) < \text{degree}(p)$ and $\text{degree}(r) < \text{degree}(z-\alpha) = 1$ so r is constant.

Now

$$0 = p(\alpha) = \overbrace{(\alpha-\alpha)}^{=0} q(\alpha) + r_0$$

so $r_0 = 0$. ~~Check: $r(z) = 0$.~~

Apply inductive assumption to factor q.

Theorem If $a_0, a_1, \ldots, a_{n-1}$ are *real* then the non-real roots of

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$$

occur in complex conjugate pairs.

Proof Suppose $p(\alpha) = 0$ with $\alpha \notin \mathbb{R}$.

Then $\overline{p(\alpha)} = \overline{0} = 0$ so

$$\overline{a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} + \alpha^n} = 0$$

so $\quad \overline{a_0} + \overline{a_1\alpha} + \cdots + \overline{a_{n-1}\alpha^{n-1}} + \overline{\alpha^n} = 0$

since $\overline{z+w} = \overline{z} + \overline{w}$

so $\quad \overline{a_0} + \overline{a_1}\,\overline{\alpha} + \cdots + \overline{a_{n-1}}(\overline{\alpha})^{n-1} + (\overline{\alpha})^n = 0$

since $\overline{zw} = \overline{z}\,\overline{w}$

so $\quad a_0 + a_1\overline{\alpha} + a_2(\overline{\alpha})^2 + \cdots + a_{n-1}(\overline{\alpha})^{n-1} + (\overline{\alpha})^n = 0$

since $a_0, \ldots, a_{n-1} \in \mathbb{R}$

But the left-hand side is $p(\overline{\alpha})$, so we have shown that $p(\overline{\alpha}) = 0$.

That is, if $\alpha \notin \mathbb{R}$ is a root of p, then so is $\overline{\alpha}$. //

Complex Functions

We write $f: \mathbb{C} \to \mathbb{C}$ for a function f which assigns a complex number $f(z)$ to every complex number z.

Examples

1. polynomials
2. complex conjugation $f(z) = \bar{z}$. $\quad z \mapsto \bar{z}$
3. modulus $f(z) = |z|$. $\quad z \mapsto |z|$
4. principal argument $\mathrm{Arg}(z) \in (-\pi, \pi]$
5. real and imaginary part

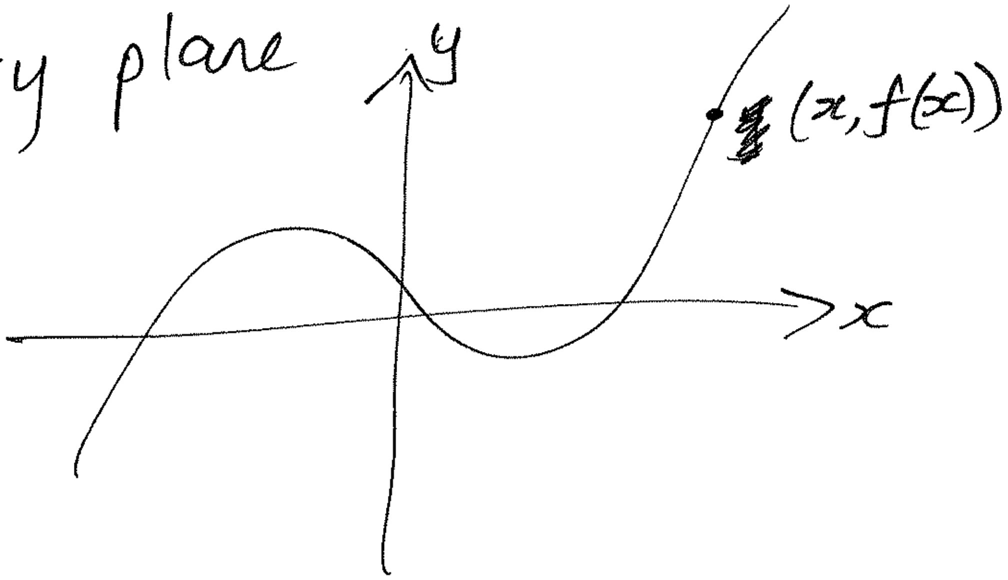
$$\mathrm{Re}(z) = x \quad \text{and} \quad \mathrm{Im}(z) = y$$

where $z = x + iy$.

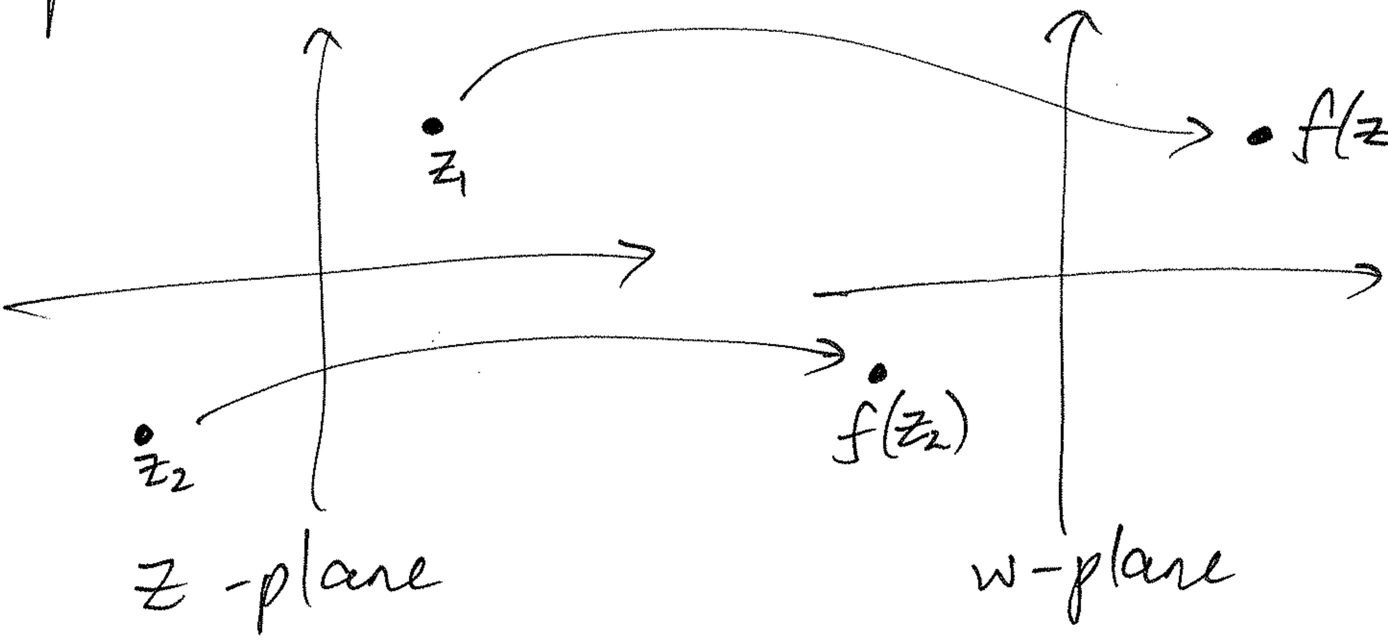
Notation If $f: \mathbb{C} \to \mathbb{C}$ with $f(z) = w$ we may write $z \longmapsto w$ and say "z maps to w".

Graphing complex functions

If $f: \overset{x}{\mathbb{R}} \to \overset{y=f(x)}{\mathbb{R}}$ we can graph f using the xy plane



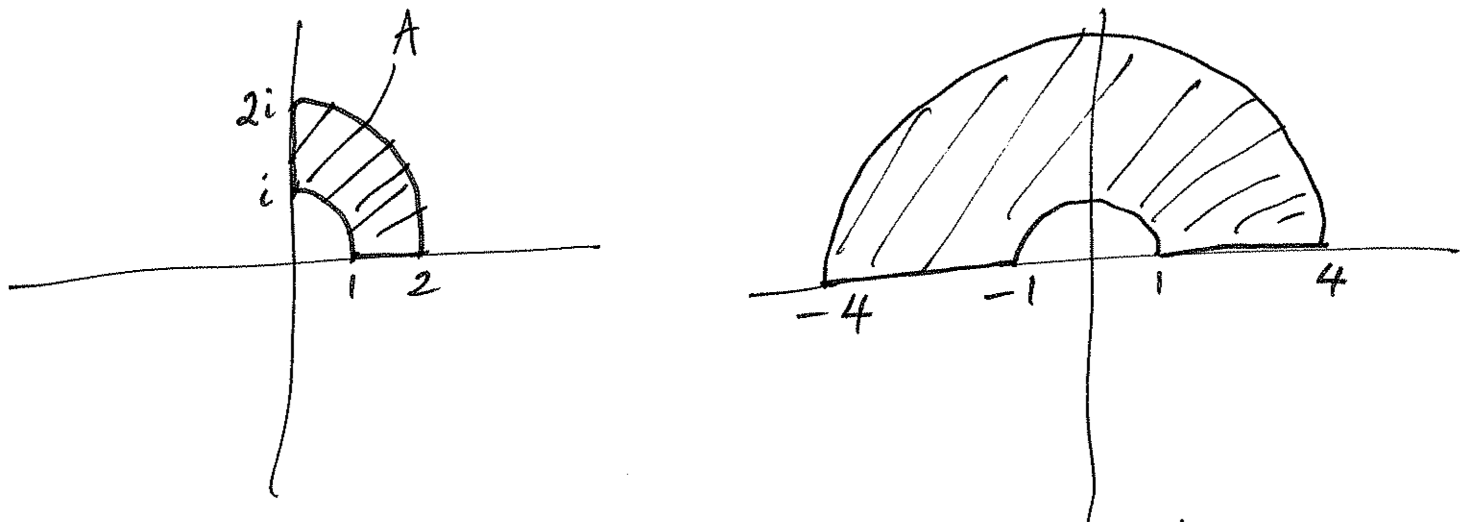
If instead $f: \mathbb{C} \to \mathbb{C}$ we can use 2 copies of the complex plane. If $f(z) = w$



More usefully, we can consider subsets of the z-plane and see what happens to them when we apply f.

<u>Example</u> $f(z) = z^2$

$$A = \{ z \in \mathbb{C} \mid 1 \le |z| \le 2, \ 0 \le \mathrm{Arg}(z) \le \tfrac{\pi}{2} \}$$



z-plane w-plane

The <u>image</u> of A is

$$\{ f(z) \mid z \in A \}$$

is $\{ w \mid 1 \le |w| \le 4, \ 0 \le \mathrm{Arg}(w) \le \pi \}$

Why? Put $z = r\,\mathrm{cis}\,\theta$
As $r = |z|$ goes from 1 to 2, $|w| = |z^2| = r^2$ goes from to 1 to 4

$$\boxed{z^2 = r^2 \,\mathrm{cis}\, 2\theta}$$

As θ goes from 0 to $\frac{\pi}{2}$, 2θ goes from 0 to π.