Week 2 Lecture 2

(1) If $a, b \in \mathbb{Z}$ then there exist $q, r \in \mathbb{Z}$ with $0 \leq |r| < |b|$ such that

$$a = bq + r$$

(2) If $a(z)$ and $b(z)$ are polynomials over $\mathbb{C}$ then there exist polynomials $q(z), r(z)$ such that $0 \leq \deg(r) < \deg(b)$ and

$$a(z) = b(z)q(z) + r(z).$$

<u>Lemma</u> (used in proof of Corollary to Fund Thm of Algebra in prev. lecture)

If $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ a polynomial over $\mathbb{C}$ has root α then $p(z) = (z-\alpha)q(z)$ (i.e. remainder term is 0) where $q(z)$ is a polynomial over $\mathbb{C}$ and $\deg(q) = n-1$.

<u>Proof</u> By (2), $p(z) = \overbrace{(z-\alpha)}^{b(z)}q(z) + r(z)$ where $0 \leq \deg(r) < \deg(z-\alpha) = 1$.

Thus $r(z)$ is a constant ie.

$$r(z) = r_0 \in \mathbb{C}.$$

Thus

$$p(z) = (z-\alpha)\, q(z) + r_0 .$$

Plug in $z=\alpha$, then $p(\alpha) = 0$ so

$$0 = \underbrace{(\alpha-\alpha)}_{=0}\, q(\alpha) + r_0$$

so we conclude $r_0 = 0$.

Thus

$$p(z) = (z-\alpha)\, q(z). \quad //$$

The complex exponential function

<u>Definition</u> Let $z = x+iy \in \mathbb{C}$.
The complex exponential function

$$\exp : \mathbb{C} \longrightarrow \mathbb{C}$$

is defined by

$$\exp(z) := \underbrace{e^x(\cos y + i \sin y)}_{\text{a complex number in polar form}}$$

We often write e^z for $\exp(z)$.

Motivation NOT RIGOUROUS! (YET)

For $x \in \mathbb{R}$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Replace x in e^x by iy

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \cdots$$

$$= 1 + iy - \frac{y^2}{2} - i\frac{y^3}{3!} + \frac{y^4}{4!} - \cdots$$

$$= \left(1 - \frac{y^2}{2} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)$$

$$= \cos y + i \sin y$$

then $e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$.

Properties

1. If $z = x$ is real, then

$$e^z = e^{x+0i} = e^x(\cos 0 + i \sin 0)$$

$$= e^x$$

$\cancel{= e^z}$ ~~since~~

i.e. the complex exponential function when restricted to $\mathbb{R}$ agrees with $x \mapsto e^x$

2. If $z = iy = 0 + iy$

$$e^z = e^0(\cos y + i\sin y)$$

so $e^{iy} = \cos y + i \sin y$ Euler's formula

In particular, put $y = \pi$

$$e^{i\pi} = \cos\pi + i\sin\pi$$

$$e^{i\pi} = -1$$

$$e^{i\pi} + 1 = 0.$$

3. $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$

Why? Let $z_1 = a + ib$, $z_2 = c + id$

then $z_1 + z_2 = a + c + i(b+d)$

so
$$\begin{aligned}\exp(z_1+z_2) &= e^{a+c}(\cos(b+d) + i\sin(b+d)) \\ &= e^a e^c(\cos b + i\sin b) \times (\cos d + i\sin d) \\ &= e^a(\cos b + i\sin b) \times e^c(\cos d + i\sin d) \\ &= \exp(z_1)\exp(z_2).\end{aligned}$$

In other words

$$e^{z_1+z_2} = e^{z_1}e^{z_2}$$

4. $|e^z| = e^x$, y is the argument of e^z (where $z = x+iy$)

5. Polar form can now be written

$$z = re^{i\theta}$$

since $e^{i\theta} = \cos\theta + i\sin\theta$.

6. $\exp(z+2\pi i) = \exp(z)$ since by 3.

$$\begin{aligned} \exp(z+2\pi i) &= e^z e^{2\pi i} \\ &= e^z(\cos 2\pi + i\sin 2\pi) \\ &= e^z \\ &= \exp(z). \end{aligned}$$

Hence

$$\exp(z+\alpha) = \exp(z)$$

whenever $\alpha = 2\pi k i$ with $k \in \mathbb{Z}$.

(compare to real exponential).

7. $\{|e^z| \mid z \in \mathbb{C}\}$

$= \{e^x \mid x \in \mathbb{R}\}$

$= \{\text{positive real numbers}\}$

In particular, $e^z \neq 0$ for any $z \in \mathbb{C}$.

8. $\{ \operatorname{Arg}(e^z) \mid z \in \mathbb{C} \}$

$= \{$ values of $\arg(e^z)$ which lie in $(-\pi, \pi] \}$

$= \{ y \mid y \in (-\pi, \pi] \}$

$= (-\pi, \pi]$.

9. By 7. and 8.

$$\{ e^z \mid z \in \mathbb{C} \} = \{ w \in \mathbb{C} \mid w \neq 0 \}$$

<u>Example</u> Find all solutions of

$$e^z = 1.$$

Put $z = x + iy$ then

$$e^z = e^x (\cos y + i \sin y)$$

and $$1 = 1 (\cos 0 + i \sin 0)$$

Hence $$e^x = 1 \quad \text{so } x = 0$$

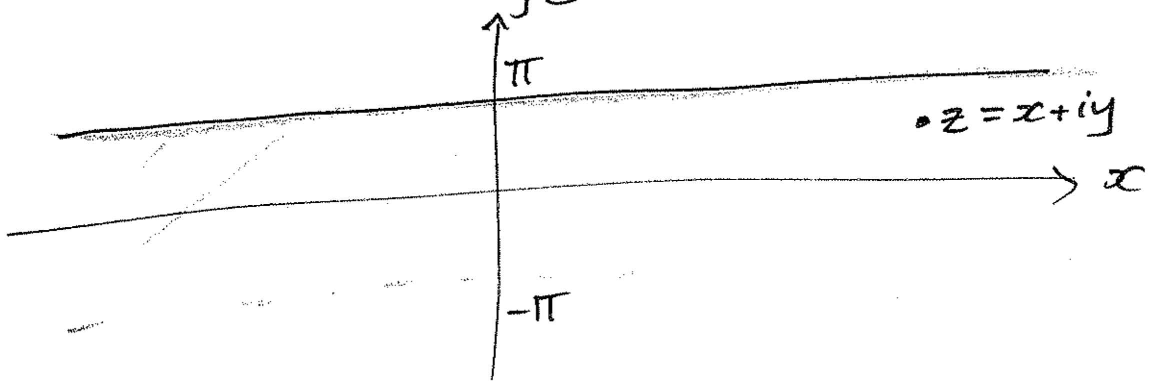
and $$y = 0 + 2k\pi, \quad k \in \mathbb{Z}.$$

The infinitely many solutions to $e^z = 1$ are $\{ 2k\pi i \mid k \in \mathbb{Z} \}$.

Visualising $z \mapsto e^z$

Examples Let

$$A = \{ z = x+iy \mid -\pi < y \leq \pi \}$$



What is the image of A under $z \mapsto e^z$?

i.e. what is $\{ e^z \mid z \in A \}$?

~~If $z \in A$ the~~

For all $z \in \mathbb{C}$, $z = x+iy$,

$$|e^z| = e^x.$$

So $\{ |e^z| \mid z \in A \}$

$= \{ e^x \mid x \in \mathbb{R} \} = \{$all positive reals$\}$

If $z \in A$ then

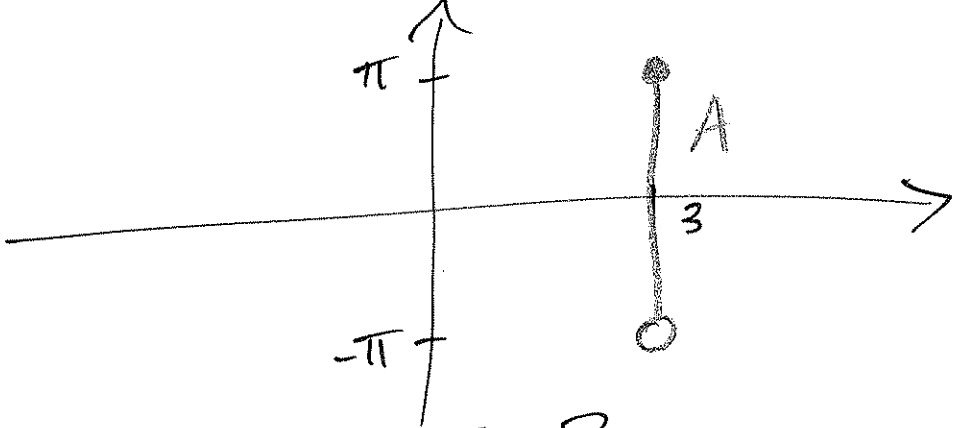
$$\operatorname{Arg}(e^z) = y$$

since ~~we test~~ the values of y in A are the same as the interval $(-\pi, \pi]$ for Arg.

Thus the image of A is all of $\mathbb{C}$ except 0.

2. What is the image of

$$A = \{ z = x+iy \mid -\pi < y \le \pi,\ x = 3 \}$$



under $z \mapsto e^z$?

Modulus? If $z \in A$ then

$$|e^z| = e^x = e^3$$

Argument? $\{ \mathrm{Arg}(e^z) \mid z \in A \} = (-\pi, \pi]$

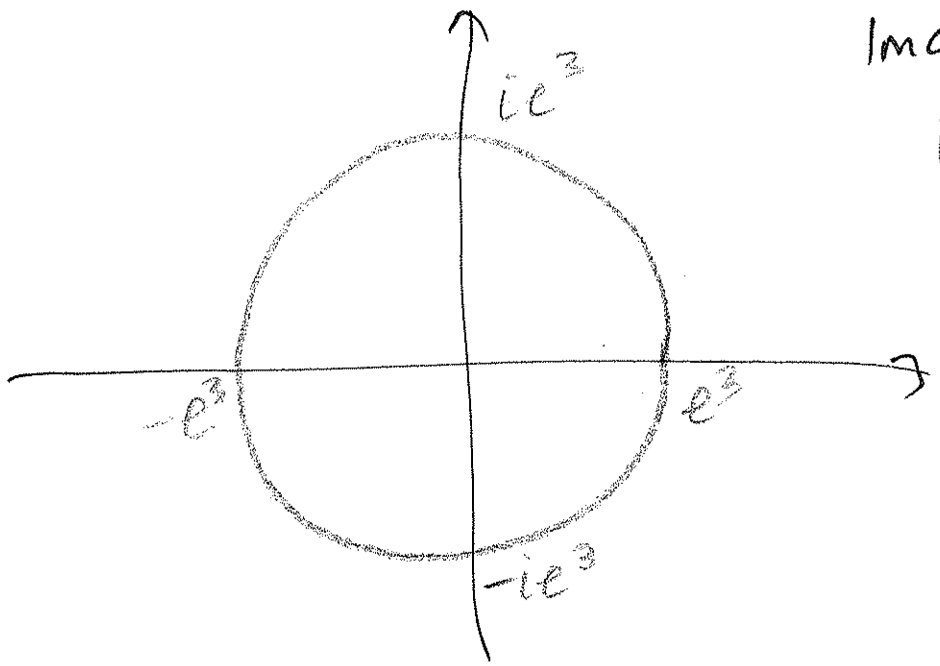


Image of A is circle centre 0 radius e^3

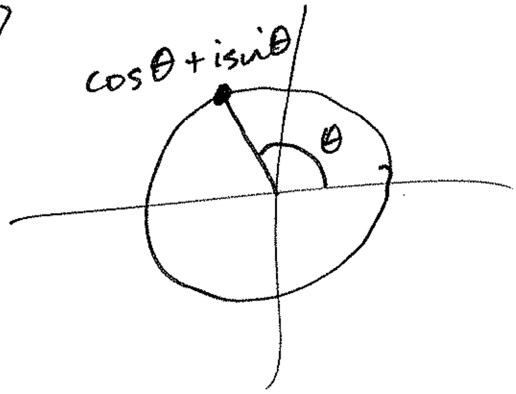
3. Let A be the unit circle.

What is image of A under $z \mapsto e^z$?

If $z \in A$ then $z = e^{i\theta} = \cos\theta + i\sin\theta$

for $-\pi < \theta \leq \pi$.

$z \mapsto e^z$ is read as "z maps to e^z"; it's the function taking z to e^z.



A

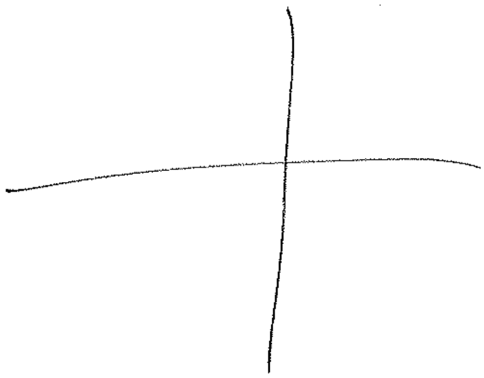


image of A

See Jenny Henderson's lecture notes.