A

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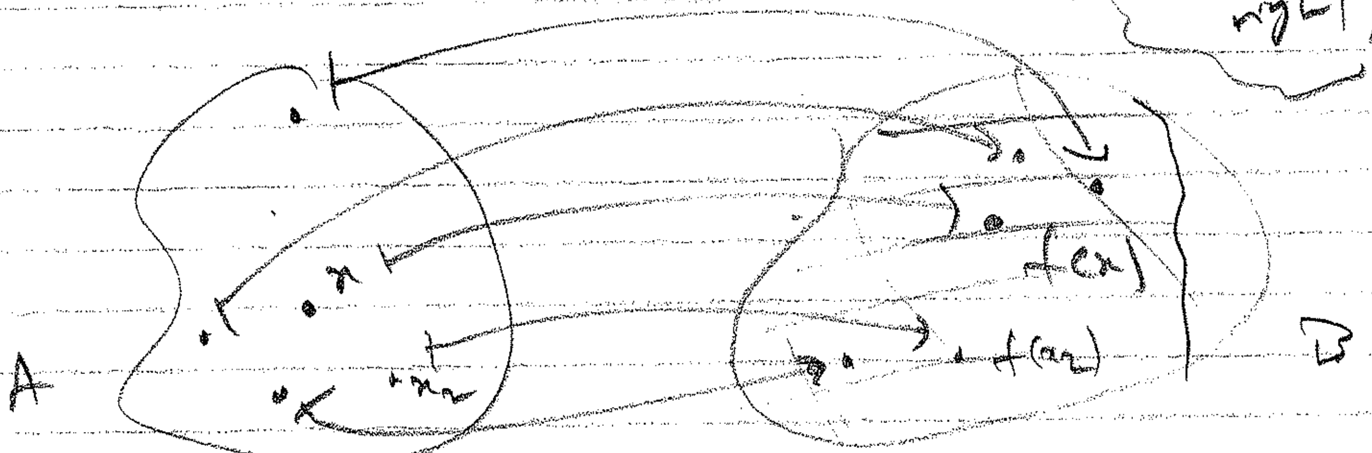
MATH 1901 — Week 3 : lecture 1 — Thurs 21/8/2013

Functions

General concept, A, B any sets

$f : A \to B$ is a rule or process that takes inputs from A and produces outputs in B

(important to think of f as an object in its own right)



codomain

domain

inputs

outputs

$$x \longmapsto f(x)$$

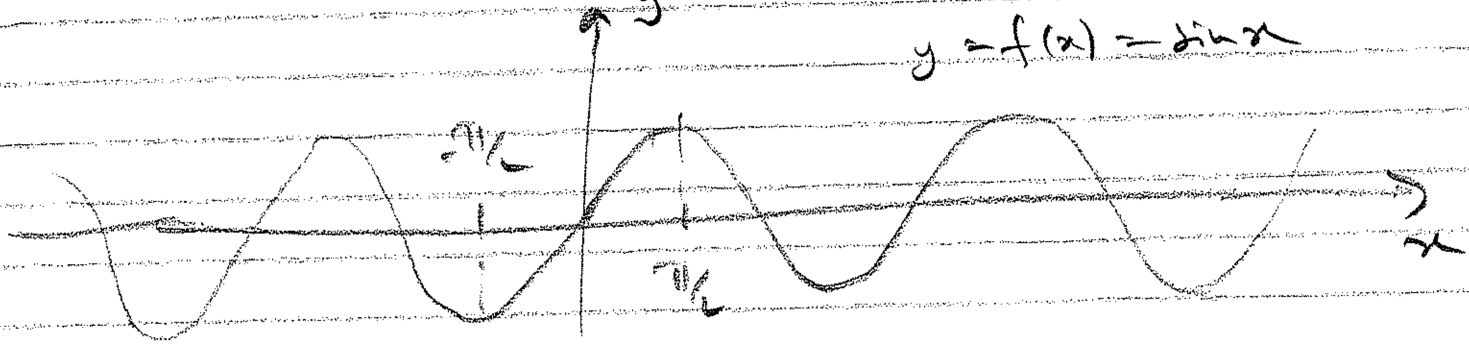
$$\text{range} = \{ f(x) \mid x \in A \} = f(A)$$

"such that"

$\boxed{\text{$f(A)$ may be \underline{much smaller} than B, the ambient space of potential outputs}}$

(3)

Circular functions (familiar trigonometry)

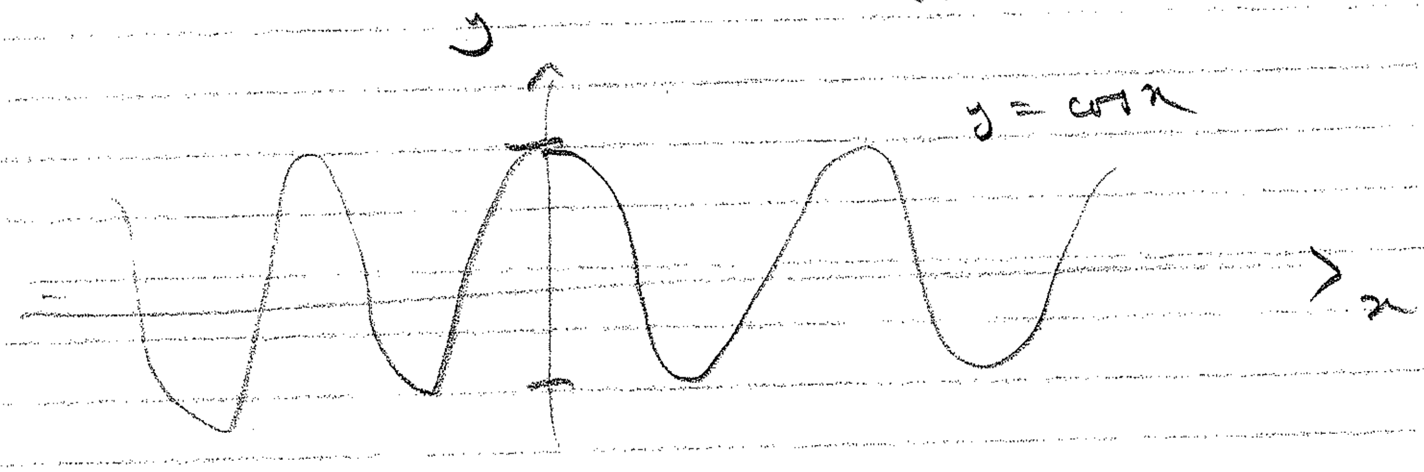


objects $\begin{cases} \sin : \mathbb{R} \to \mathbb{R}, & x \mapsto \sin x \\ \cos : \mathbb{R} \to \mathbb{R}, & x \mapsto \cos x \end{cases}$

domain = codomain = $\mathbb{R}$, range = $[-1, 1]$
in each case

$$\boxed{\cos x = \sin(x + \pi/2), \quad \sin x = \cos(x - \pi/2)}$$

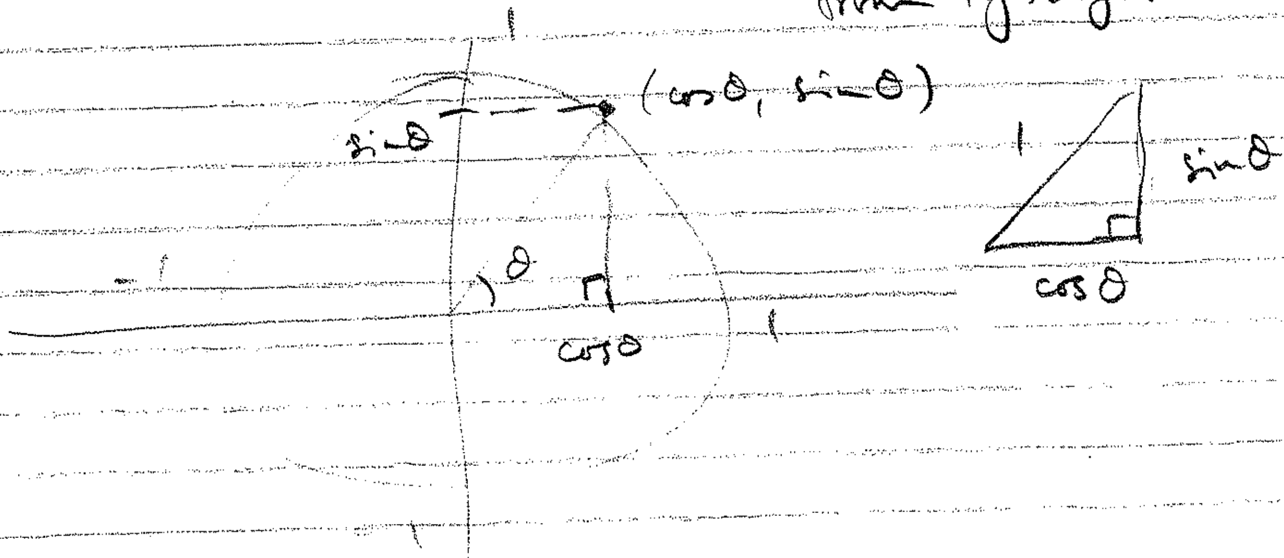
graphs obtained by
mutually inverse horizontal
translations



c

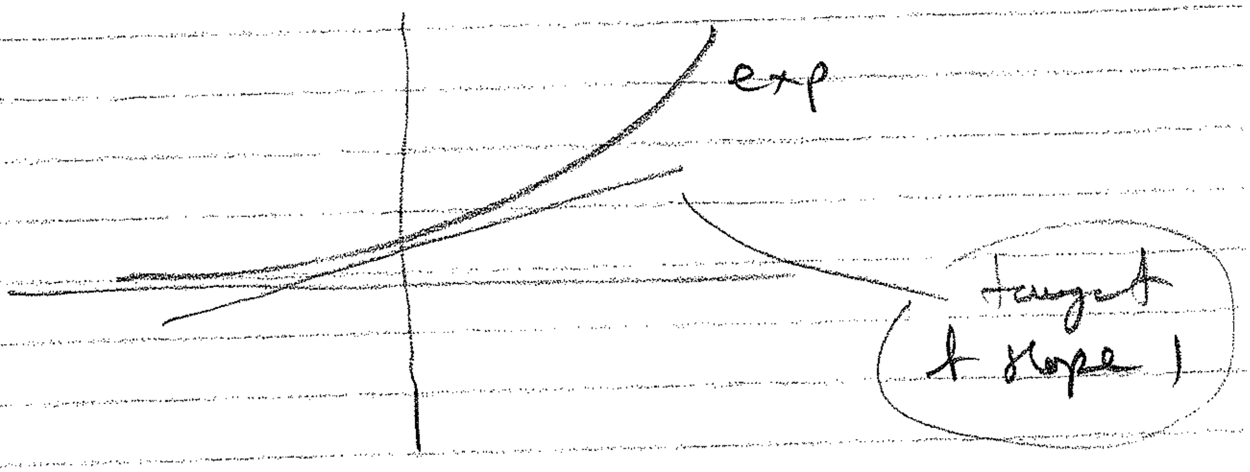
circular identity : $\boxed{\cos^2 x + \sin^2 x = 1}$

from Pythagoras



Exponential function (also familiar) :

$\exp : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \exp(x) = e^x$



domain = codomain = $\mathbb{R}$, range = $(0, \infty)$

We know exp, sin, cos are related in complex function theory.

D

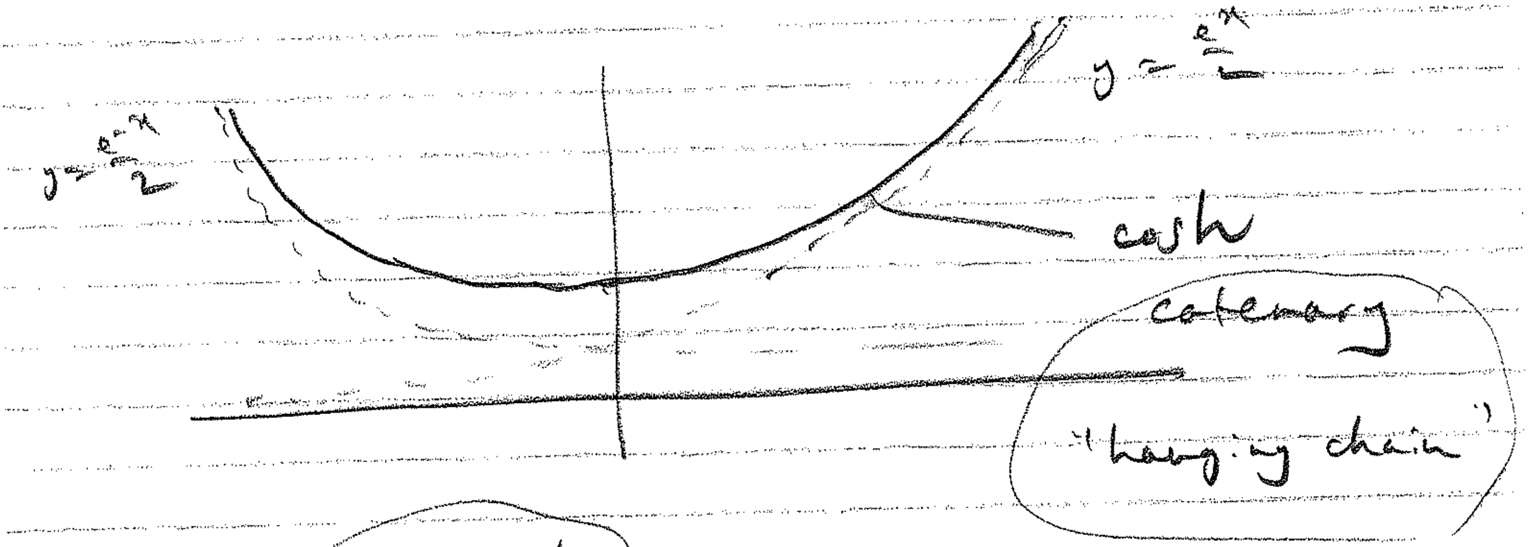
Hyperbolic functions (unfamiliar?)

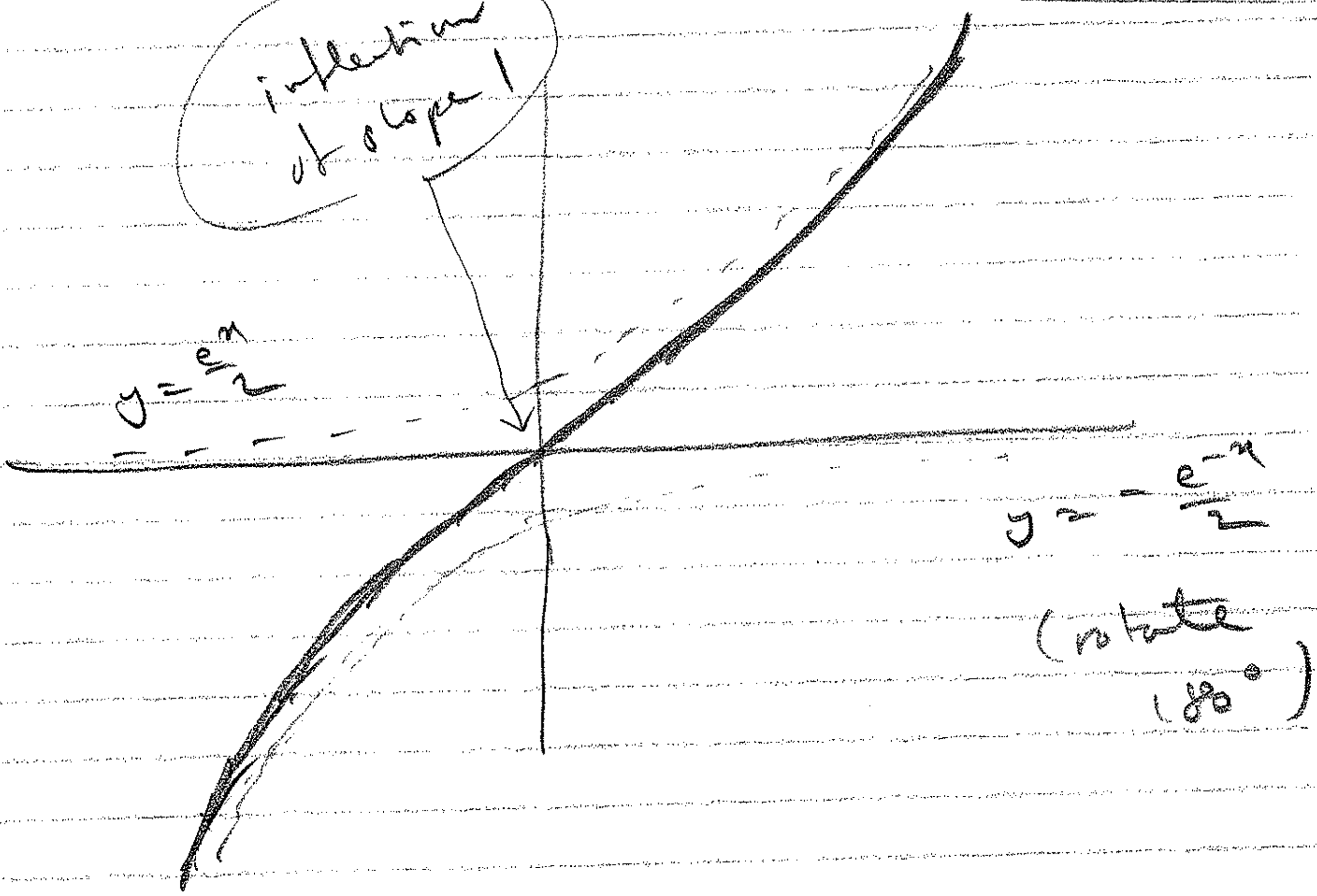
Define

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

h for hyperbolic





E

$\sinh, \cosh : \mathbb{R} \to \mathbb{R}$

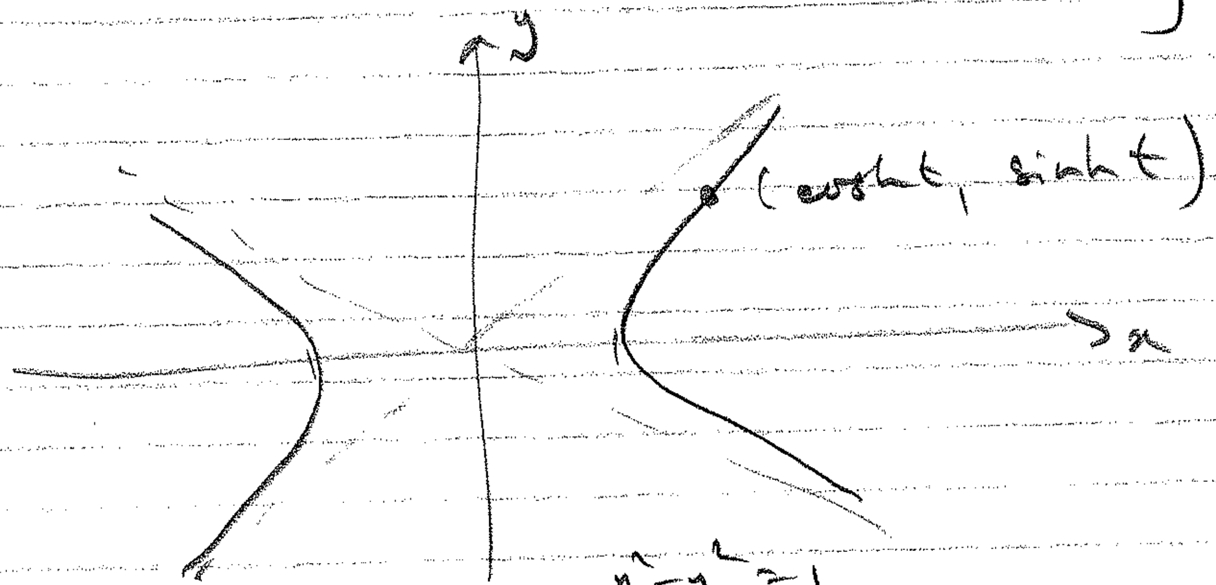
range of $\cosh = [1, \infty)$

range of $\sinh = \mathbb{R} =$ codomain.

<u>Easy to check</u>:

$$\boxed{\cosh^2 x - \sinh^2 x = 1}$$

<u>hyperbolic</u> identity



$x^2 - y^2 = 1$
hyperbola
with asymptotes
$y = \pm x$

F

Well-behaved derivatives

$$e^x \mapsto e^x \qquad \text{(period 1)}$$

$$\sin x \mapsto \cos x \mapsto -\sin x \mapsto -\cos x \mapsto \sin x$$

(period 4)

$$\sinh x \mapsto \cosh x \mapsto \sinh x$$

(period 2)

All are indestructible under differentiation, unlike polynomial functions, which can be made to vanish by enough differentiation.

How can we build an indestructible polynomial??

$$0 \leftarrow 1 \leftarrow x \leftarrow \frac{x^2}{2} \leftarrow \frac{x^3}{3(2)} \leftarrow \frac{x^4}{4!} \leftarrow \frac{x^5}{5!} \leftarrow \cdots$$

keep going forever

Now put

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$$

forever

then $f'(x) = f(x)$ (period 1)

and in fact $f(x) = e^x$.

Taylor series
see later

Try summing every second term:

$$g(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \quad \text{(even powers)}$$

$$h(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots \quad \text{(odd powers)}$$

Then $\boxed{g'(x) = h(x), \quad h'(x) = g(x)}$

(period 2)

Which shall be cosh, sinh??

cosh is even (reflectional symmetry)

sinh is odd (rotational ")

and indeed $\boxed{g(x) = \cosh x, \quad h(x) = \sinh x}$

How to get period 4?

Play with alternating signs: (even powers)

Put $k(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$$l(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad \text{(odd powers)}$$

cos is even, sin is odd

and indeed $\boxed{k(x) = \cos x, \quad l(x) = \sin x}$

(4)

Function notation is general and can involve weird sets

e.g. $A = \{\text{people in this room}\}$

$B = \{\text{seats in this room}\}$

$f: A \to B$, $x \longmapsto$ seat x is sitting on.

Call f <u>onto</u> (surjective) if $f(A) = B$, which would mean every seat is occupied (Not so today, so f is not quite onto)

Call f <u>one-one</u> (<u>injective</u>) if different people occupy different seats

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

True today (and most days)

If f were both one-one & onto then it would be easy to count the number of people, since seats are well organised.

(I)

Principle in combinatorics :

Try to find one-one correspondences (injective & surjective = bijective) maps between a "wild" set A and a "tame" set B and then you know

$$|A| = |B| . \quad \text{(same size)}$$

When sets have structure & bijections exist that "preserve" structure then you have a powerful means of classifying interesting mathematical phenomena and taming wild beasts.