A

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MATH1901 Week 3 : lecture 2 Fri 22/3/13

Functions continued

Operations on functions

Arithmetic, let $f, g : A \to \mathbb{R}$

has arithmetic

any ring arithmetic will do

Define $f \pm g$, fg, $f/g : A \to \mathbb{R}$ by

$f \pm g\,(x) = f(x) \pm f(g)$, $fg\,(x) = f(x)g(x)$

$f/g\,(x) = f(x)/g(x)$ $\forall x \in \mathbb{R}$

assuming $g(x) \neq 0$ $\forall x$

e.g. $\sinh, \cosh : \mathbb{R} \to \mathbb{R}$

$$(\sinh + \cosh)(x) = \frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} = e^x$$

so $\boxed{\sinh + \cosh = \exp}$

$$\frac{\sinh}{\cosh}(x) = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh x$$

$\boxed{\frac{\sinh}{\cosh} = \tanh}$ analogous to $\boxed{\frac{\sin}{\cos} = \tan}$

e.g. $f : \mathbb{Z}^+ \to \mathbb{R}$ can be identified with a sequence

$$f = \begin{array}{ccccccc} (x_1, & x_2, & x_3, & \ldots, & x_n, & \ldots) \\ \| & \| & \| & & \| & \\ f(1) & f(2) & f(3) & & f(n) & \ldots \end{array}$$

and obtain an arithmetic of sequences.

<u>Composition</u>: $f : A \to B$, $g : B \to C$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, bend left, "g"] & C \end{tikzcd}$$

Define

$$g \circ f : A \to C \text{ by } g \circ f(x) = g(f(x)) \quad \forall x \in A$$

<u>Example</u> (motivated by hyperbolae!!)

Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{1\}$

$$x \mapsto f(x) = \underbrace{\frac{x+1}{x}}_{\text{rational fcn}} = 1 + \frac{1}{x}$$

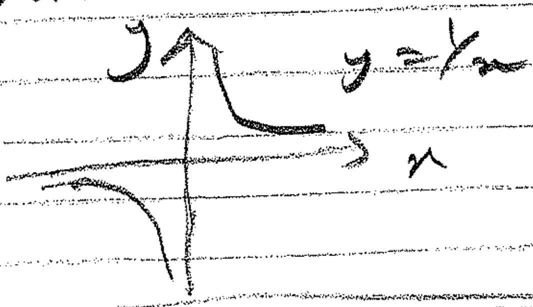
$g : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{0\}$

$$x \mapsto g(x) = \frac{1}{x-1}$$

arithmetic from before

(c)

Their graphs are vertical & horizontal displacements of



$$\begin{tikzcd} \mathbb{R}\setminus\{0\} \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & \mathbb{R}\setminus\{1\} \arrow[r, "g"] & \mathbb{R}\setminus\{0\} \end{tikzcd}$$

$$g\circ f\,(x) = g(f(x)) = g\left(1+\tfrac{1}{x}\right) = \frac{1}{1+\frac{1}{x}-1} = x$$

$$f\circ g\,(x) = f(g(x)) = f\left(\frac{1}{x-1}\right) = 1 + \frac{1}{\frac{1}{x-1}}$$

$$= 1+x-1 = x.$$

$$\begin{tikzcd} \mathbb{R}\setminus\{0\} \arrow[r, bend left, "f"] \arrow[loop below, "g\circ f"] & \mathbb{R}\setminus\{1\} \arrow[l, bend left, "g"] \arrow[loop below, "f\circ g"] \end{tikzcd}$$

$g\circ f$ identity map on $\mathbb{R}\setminus\{0\}$

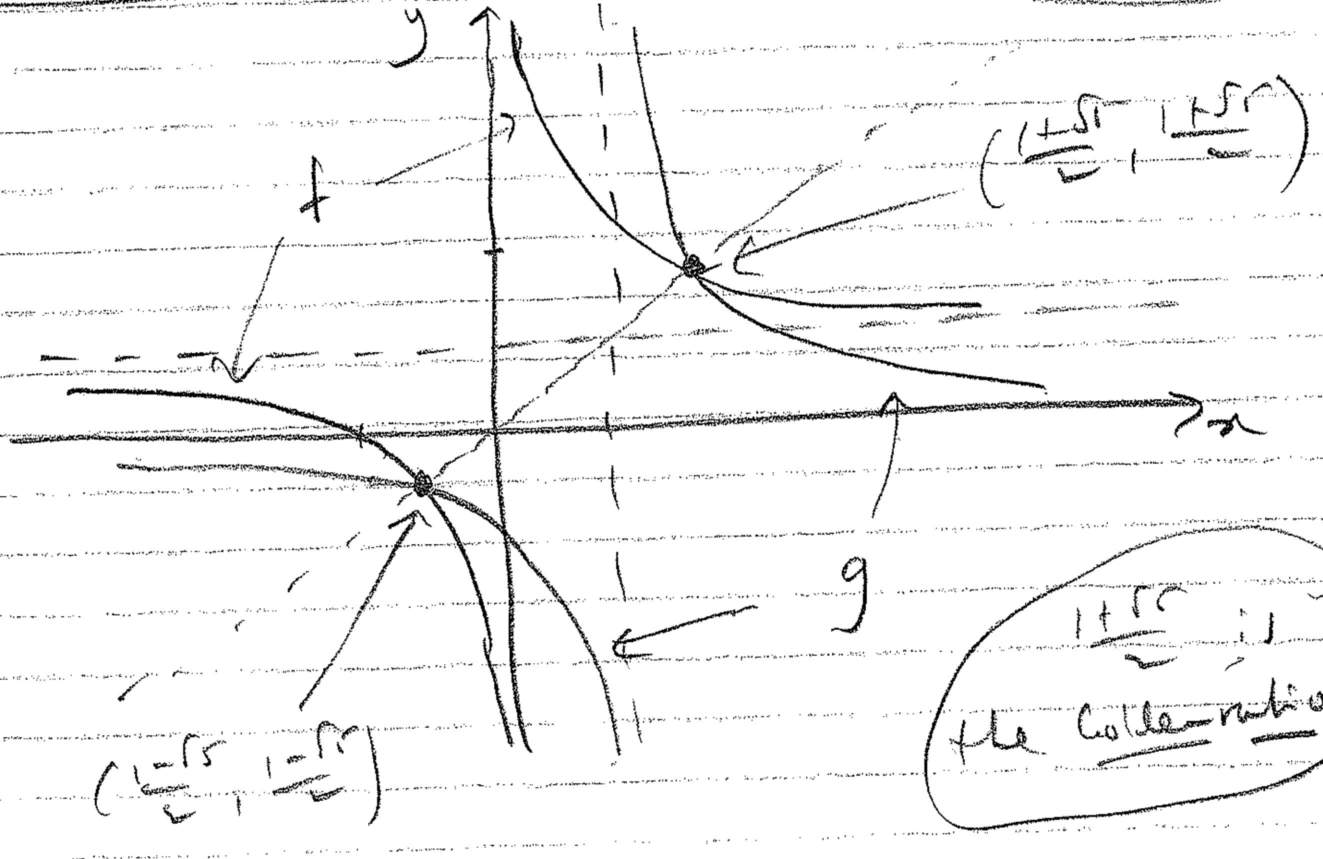
$f\circ g$ identity map on $\mathbb{R}\setminus\{1\}$

$\boxed{f\circ g \neq g\circ f}$ only just though rules are the same

D

f and g undo other

reflected in their graphs, obtained from each other by reflection in line $y = x$



$\frac{1+\sqrt{5}}{2}$ is the Golden ratio

domain f = range g = $\mathbb{R} \setminus \{0\}$

range f = domain g = $\mathbb{R} \setminus \{1\}$

Notice graphs of f, g satisfy both the horizontal & vertical line tests

special property

automatic for any function

E

Call $f: A \to B$ <u>onto</u> (<u>surjective</u>) if

$f(A) = B$, i.e. $(\forall y \in B)(\exists x \in A)\ f(x) = y$.

Call $f: A \to B$ <u>one-one</u> (<u>injective</u>) if

different inputs produce different outputs,

i.e. $(\forall x_1, x_2 \in A)\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$

equivalently

$(\forall x_1, x_2 \in A)\ f(x_1) = f(x_2) \implies x_1 = x_2$

$P \Rightarrow Q$ equivalent to $\underbrace{\sim Q \Rightarrow \sim P}_{\text{contrapositive}}$

Call f <u>bijective</u> if f is one-one & onto

in which case the <u>inverse</u> function exists

$$f^{-1} : B \to A$$

given by indirect rule

$$y = f(x) \iff x = f^{-1}(y)$$

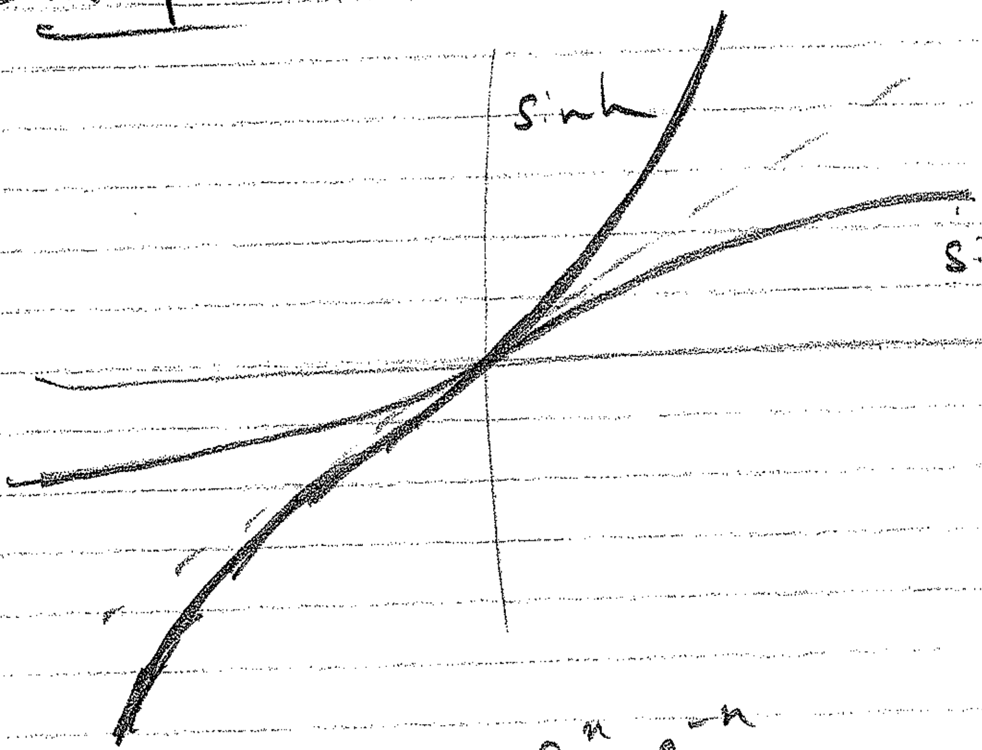
F

$$\begin{tikzcd} x \arrow[r, bend left, "f"] & y \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

"undo" each other

In previous example $\boxed{g = f^{-1},\ f = g^{-1}}$

<u>Example</u>: Find f^{-1} when $f = \sinh$



Put $y = \sinh x = \dfrac{e^x - e^{-x}}{2}$ so $2y = e^x - e^{-x}$,

so $2ye^x = e^{2x} - 1$, so $e^{2x} - 2ye^x - 1 = 0$,

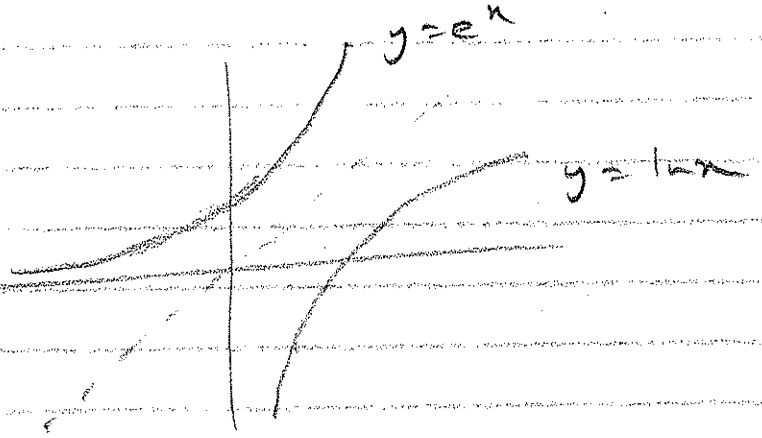
so $e^x = \dfrac{2y \pm \sqrt{4y^2+4}}{2} = y \pm \sqrt{y^2+1}$

so $\boxed{x = \ln\left(y + \sqrt{y^2+1}\right)}$ ignoring negative solution

Hence $\sinh^{-1}(y) = \ln(y+\sqrt{y^2+1})$

so $\boxed{\sinh^{-1}(x) = \ln(x+\sqrt{x^2+1})}$

Note: if $h(x) = e^x$ then $h^{-1}(x) = \ln x$



If x is large & positive then

$$\sinh^{-1}(x) \approx \ln(2x) = \ln x + \ln 2$$

If x is large & negative then

$$\sinh^{-1}(x) \approx \ln\left(-\frac{1}{2x}\right) = -\ln(-x) - \ln 2$$

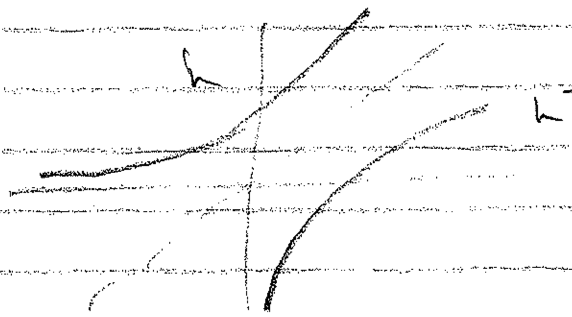
tricky exercise

(H)

Hence $\sinh^{-1}(y) = \ln\left(y + \sqrt{y^2+1}\right)$

ie. $$\boxed{\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)}$$

Note: If $h(x) = e^x$ then $h^{-1}(x) = \ln x$



vertical displacement of $\ln$ graph

As $x \to \infty$, $\sinh^{-1}(x) \simeq \ln(2x) = \ln x + \ln 2$

because $\sqrt{x^2+1} \simeq x$

By rotational symmetry, as $x \to -\infty$

$$\sinh^{-1}(x) \simeq -\ln(-x) - \ln 2$$

Check (conjugation trick): for $x \to -\infty$

$$\ln\left(x + \sqrt{x^2+1}\right) = \ln\left(\frac{\left(x+\sqrt{x^2+1}\right)\left(x - \sqrt{x^2+1}\right)}{x - \sqrt{x^2+1}}\right)$$

$$= \ln\left(\frac{x^2 - (x^2+1)}{x - \sqrt{x^2+1}}\right) = \ln\left(\frac{-1}{x - \sqrt{x^2+1}}\right)$$

$$\simeq \ln\left(\frac{-1}{2x}\right) = -\ln(-2x) = -\ln(-x) - \ln 2$$

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