Week 4

Limits

Throughout: $f:\mathbb{R}\to\mathbb{R}$, $a\in\mathbb{R}$, $l\in\mathbb{R}$

Notation

$$\lim_{x\to a} f(x) = l \quad \text{or} \quad f(x)\to l \text{ as } x\to a$$

is read as "the limit of $f(x)$ as x goes to a is l" or "$f(x)$ goes to l as x approaches a".

Informal definition As x gets close to a, $f(x)$ gets close to l.

Modifying this to get a formal definition:

Step 1 We want $f(x)$ to take values arbitrarily close to l.

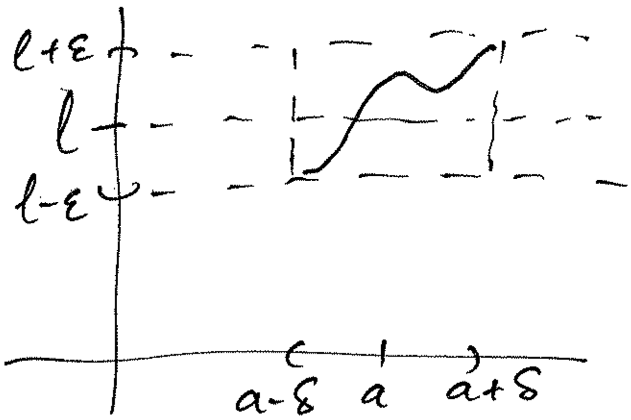
That is, we want $\underbrace{|f(x)-l|}_{\text{distance between } f(x) \text{ and } l}$ to be arbitrarily small.

That is, we want $|f(x)-l|$ to
be smaller than any positive number.
That is, for every $\varepsilon > 0$

Greek letter epsilon

we want $|f(x)-l| < \varepsilon$.

Step 2



Given $\varepsilon > 0$, we want $|f(x)-l| < \varepsilon$ whenever x is close enough to a. (but not equal to a)

That is, for all $\varepsilon > 0$, there is a $\delta > 0$ so that whenever

Greek letter delta

$$0 < \underbrace{|x-a|}_{\text{distance between } x \text{ and } a} < \delta$$

we have

$$|f(x)-l| < \varepsilon.$$

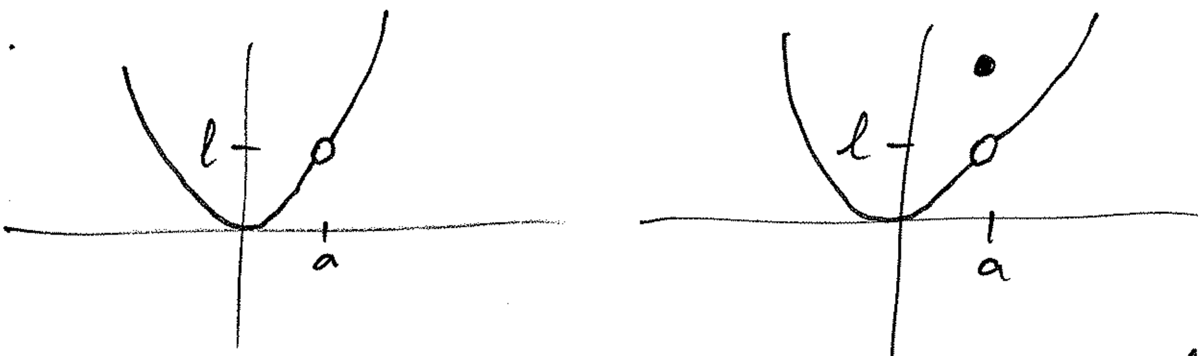
Definition

The limit of $f(x)$ as x approaches a exists and equals l if for all $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < |x - a| < \delta$ we have $|f(x) - l| < \varepsilon$.

(We can make $f(x)$ as close as we like to l by making x close enough to a.)

<u>Note</u> If $\lim_{x \to a} f(x) = l$ then f may or may not be defined at $x = a$, and we could have $\lim_{x \to a} f(x) = l$ but $f(a) \neq l$.

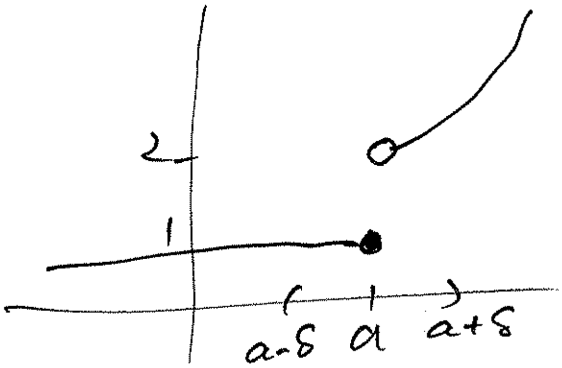
e.g.



In both situations, $\lim_{x \to a} f(x) = l$.

One-Sided Limits

The <u>right-hand limit</u> $\lim_{x \to a^+} f(x) = \ell$



if for all $\varepsilon > 0$ there is a $\delta > 0$ so that whenever

$$0 < x - a < \delta$$

(i.e. $a < x < a + \delta$)

we have $|f(x) - \ell| < \varepsilon$.

The <u>left-hand limit</u> $\lim_{x \to a^-} f(x) = \ell$

if for all $\varepsilon > 0$ there is a $\delta > 0$

so that whenever

$$0 < a - x < \delta$$

(i.e. $a - \delta < x < a$)

we have $|f(x) - \ell| < \varepsilon$.

The <u>2-sided limit</u> $\lim_{x \to a} f(x) = \ell$

if and only if $\lim_{x \to a^-} f(x) = \ell$ and $\lim_{x \to a^+} f(x) = \ell$.

Examples

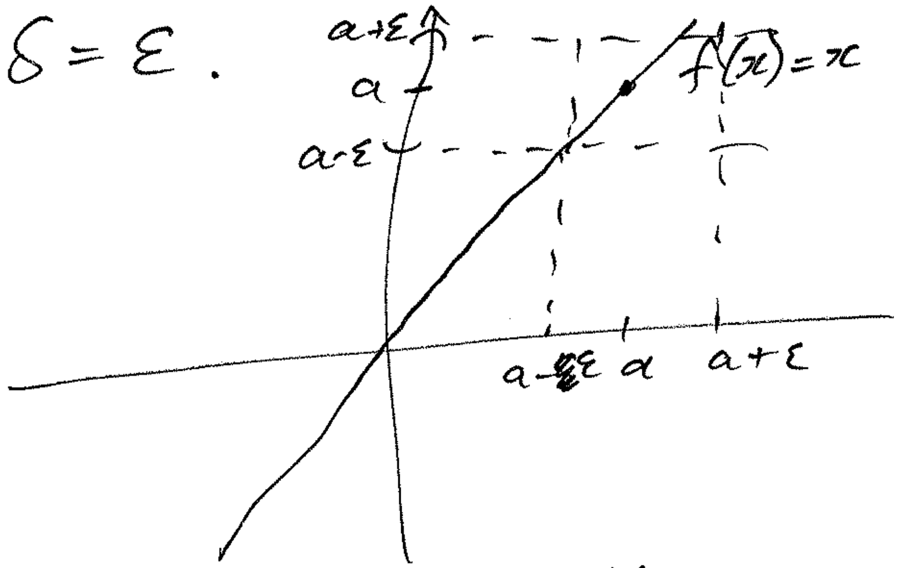
1. Let $f(x) = 4x+1$.
Given $\varepsilon > 0$ find $\delta > 0$ so that whenever $0 < |x| < \delta$ we have $|f(x) - 1| < \varepsilon$.
(Hence, $\lim_{x \to 0} f(x) = 1$. } Put $a = 0$, $l = 1$.)

Answer Let $\varepsilon > 0$.
To ensure
$$|f(x) - 1| < \varepsilon$$
$$|f(x)-1| = |4x+1-1| = |4x| = 4|x|$$
enough to have $|x| < \frac{\varepsilon}{4}$.
Put $\delta = \frac{\varepsilon}{4}$. Then if
$$0 < |x| < \delta$$
we have $|f(x) - 1| < \varepsilon$ as required.

2. Prove that
$$\lim_{x \to a} x = a.$$

Let $\varepsilon > 0$. We want to find $\delta > 0$ so that whenever
$$0 < |x - a| < \delta$$
we have
$$|x - a| < \varepsilon.$$
(Here $l = a$, $f(x) = x$.)

Take $\delta = \varepsilon$.



3. Tutorial: prove that
$$\lim_{x \to a} k = k$$
where $k \in \mathbb{R}$ is constant.

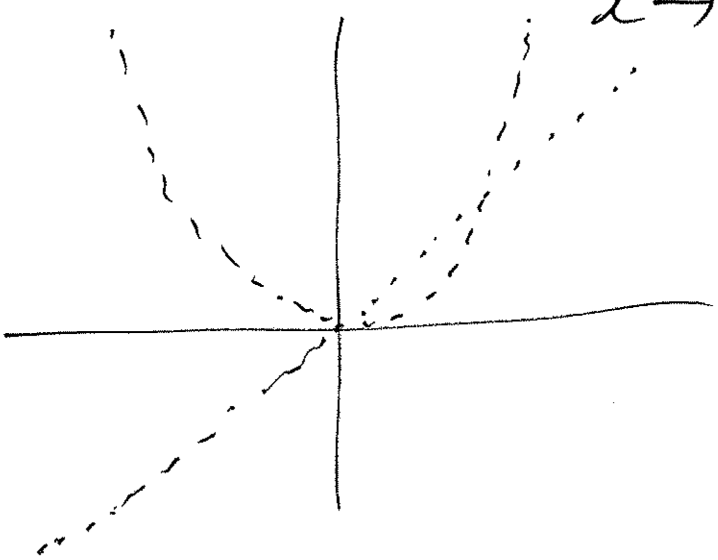
4. To show that $\lim_{x \to a} f(x)$ does not exist, i.e. that there is no real number ℓ such that $\lim_{x \to a} f(x) = \ell$, you need to show:

there is some $\varepsilon > 0$ so that for all $\delta > 0$, there is some x with $0 < |x-a| < \delta$ but

$$|f(x) - \ell| \geq \varepsilon.$$

5. Let $$f(x) = \begin{cases} x & \text{if } x \text{ rational} \\ x^2 & \text{if } x \text{ irrational} \end{cases}$$

Prove that $\lim_{x \to 0} f(x) = 0$.



Let $\varepsilon > 0$.

~~Want~~

Want $\delta > 0$ so that whenever 8

$$0 < |x - 0| < \delta \quad (a=0)$$

we have

$$|f(x) - 0| < \varepsilon. \quad (l=0)$$

To ensure $|f(x)| < \varepsilon$, ~~it's~~

~~enough to~~ we require

$$|x| < \varepsilon \ \underline{\text{and}} \ |x^2| < \varepsilon.$$

(since $f(x)$ takes values x and x^2 in <u>any</u> interval around 0).

i.e. $|x| < \varepsilon$ <u>and</u> $|x|^2 < \varepsilon$

i.e. $|x| < \varepsilon$ <u>and</u> $|x| < \sqrt{\varepsilon}$

Choose $\delta = \min(\varepsilon, \sqrt{\varepsilon})$.

Then whenever

$$0 < |x| < \delta$$

we have

$$|x| < \varepsilon \text{ and } |x| < \sqrt{\varepsilon}$$

hence

$$|x| < \varepsilon \text{ and } |x^2| < \varepsilon$$

so $|f(x)| < \varepsilon$ as required.

Limit laws (Useful for computation)

Suppose $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$.

1. Addition / subtraction

$$\lim_{x\to a}\left(f(x) \pm g(x)\right) = \left(\lim_{x\to a} f(x)\right) \pm \left(\lim_{x\to a} g(x)\right)$$
$$= l \pm m.$$

2. Products

$$\lim_{x\to a}\left(f(x)g(x)\right) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$$
$$= lm$$

Examples

1. For $k \in \mathbb{R}$

$$\lim_{x\to a} kg(x) = k\left(\lim_{x\to a} g(x)\right)$$
$$= km.$$

2. By induction, for n positive integer

$$\lim_{x\to a} x^n = a^n.$$

3. Quotients if $m \neq 0$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{l}{m}$$

4. Square roots if $l \geq 0$

$$\lim_{x \to a} \sqrt{f(x)} = \sqrt{\lim_{x \to a} f(x)} = \sqrt{l}$$

5. Squeeze law / Pinching theorem / Sandwich theorem

If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = l$ and

$$f(x) \leq h(x) \leq g(x)$$

on an interval containing a, except possibly at $x = a$, then

$$\lim_{x \to a} h(x) = l \text{ as well.}$$

e.g. $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

Limits as $x \to \pm\infty$

(horizontal asymptotes)

$\lim\limits_{x \to \infty} f(x) = \ell$ if for all $\varepsilon > 0$ there is an $M > 0$ so that whenever

$$x > M$$

we have

$$|f(x) - \ell| < \varepsilon.$$

Similarly, $\lim\limits_{x \to -\infty} f(x) = \ell$ if for all $\varepsilon > 0$ there is an $M > 0$ so that whenever

$$x < -M$$

we have

$$|f(x) - \ell| < \varepsilon.$$

Examples

1. Prove that $\lim_{x \to \infty} \frac{1}{x} = 0$.

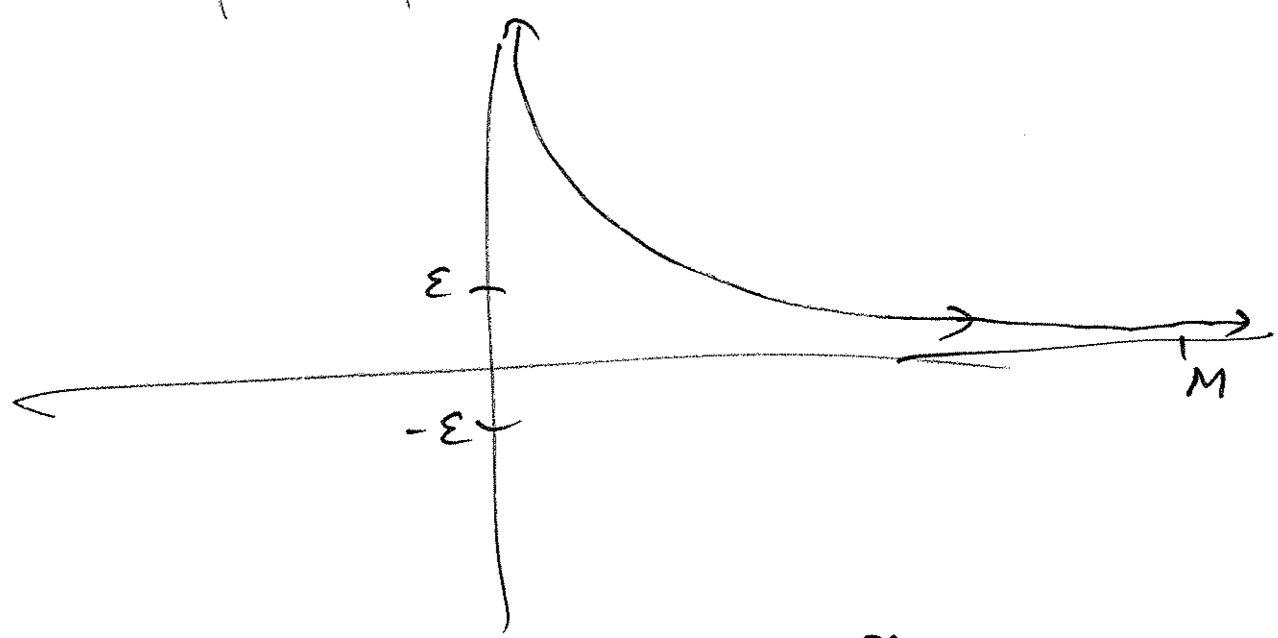
Let $\varepsilon > 0$. To ensure $\left|\frac{1}{x}\right| < \varepsilon$

$$\boxed{\begin{array}{c} f(x) = \frac{1}{x} \\ l = 0 \end{array}}$$

$$\left|\frac{1}{x}\right| = \frac{1}{|x|} < \varepsilon \iff \frac{1}{\varepsilon} < |x|$$

Choose $M = \frac{1}{\varepsilon}$.

Then whenever $x > M$, we have $\left|\frac{1}{x}\right| < \varepsilon$ as required.



2. Prove that $\lim_{x \to -\infty} e^x = 0$.