Week 5 Lecture 1

Recall: $\lim_{x \to a} f(x) = \ell$ if for all $\varepsilon > 0$ there is a $\delta > 0$ so that whenever

$$0 < |x - a| < \delta$$

we have

$$|f(x) - \ell| < \varepsilon.$$

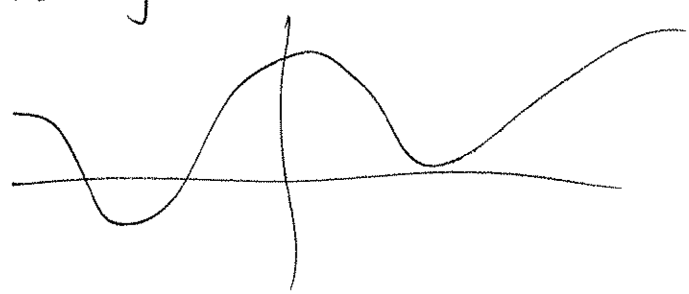
Continuity

A function f is continuous at the point a if a is in the domain of f and $\lim_{x \to a} f(x) = f(a)$.

i.e. $\lim_{x \to a} f(x)$ exists and equals $f(a)$.

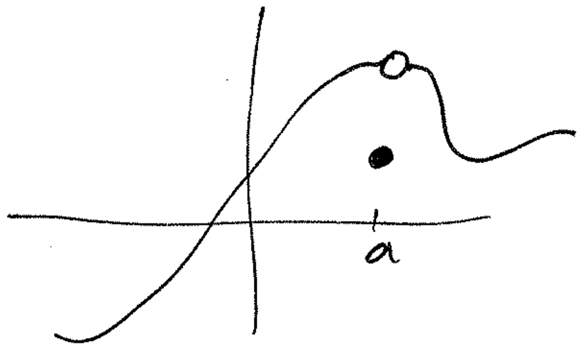
A function f is continuous if it is continuous at each point in its domain.

(Intuition: f has no breaks or jumps

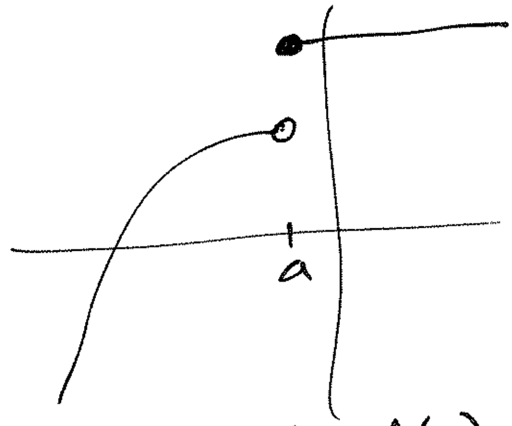


)

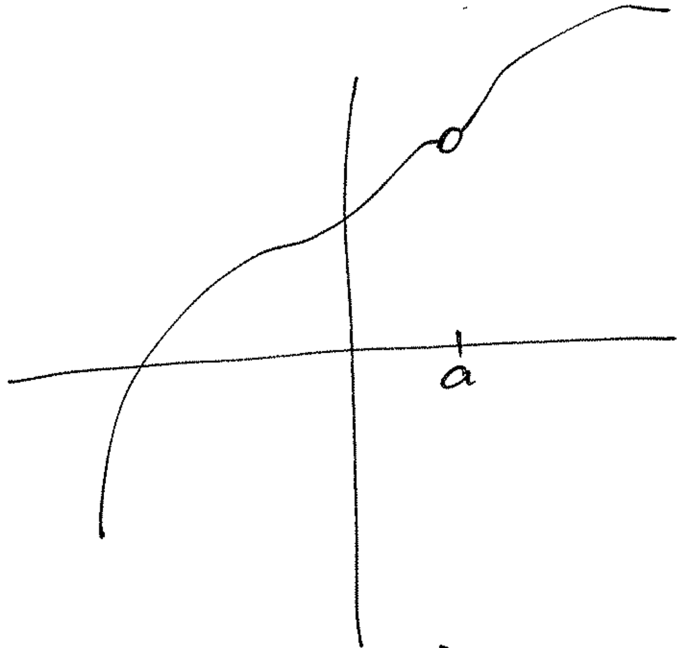
Ways for continuity to fail:



$\lim_{x \to a} f(x)$ exists
but is not equal
to $f(a)$,
so f is not
continuous at a.



here, $\lim_{x \to a} f(x)$
does not exist
(since left- and right-
hand limits are
different),
so f is not continuous
at a.



here $\lim_{x \to a} f(x)$ exists
but ~~f~~ a is not in the
domain of f so
f is not continuous
at a.

Examples of continuous functions

1. For any constant c, the function
$$f(x) = c$$
is continuous since for all $a \in \mathbb{R}$
$$\lim_{x \to a} f(x) = \lim_{x \to a} c = c = f(a).$$

2. The function $f(x) = x$ is continuous since
$$\lim_{x \to a} f(x) = \lim_{x \to a} x = a = f(a).$$

3. The function
$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$
is continuous at $x = 0$ since
$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x \sin \frac{1}{x}$$
$$= 0 \quad \text{by the Squeeze Law}$$
$$= f(0).$$

4. The functions

$$e^x, \ \ln x, \ \sin x, \ \cos x$$

are all continuous.
(Note: $\ln x$ continuous on $(0,\infty)$)

i.e. for example

$$\lim_{x \to a} \sin x = \sin a$$

for all $a \in \mathbb{R}$.

5. The function

$$f(x) = \begin{cases} 0 & x \text{ rational} \\ 1 & x \text{ irrational} \end{cases}$$

is not continuous at <u>any</u> point a since $\lim_{x \to a} f(x)$ never exists.

<u>ε-δ definition of continuity at a point</u>

If a is in the domain of f, then f is continuous at a if for all $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < |x-a| < \delta$, $|f(x) - f(a)| < \varepsilon$.

Theorem

If f and g are continuous at a point a, then so are

1. $f+g$ 2. $f-g$ 3. fg 4. $\frac{f}{g}$ provided $g(a) \neq 0$.

Proof All follow from limit laws.

e.g. 1. $\lim_{x \to a} (f+g)(x) = \lim_{x \to a} (f(x)+g(x))$

$= \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

by Additive Law for limits

$= f(a) + g(a)$

since f and g are continuous at a.

$= (f+g)(a)$.

thus $f+g$ is continuous at a.

3. $(fg)(x) = f(x)g(x)$

4. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

Examples

1. Polynomials are continuous.

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

$$a_i \in \mathbb{R}$$

~~sum of~~

2. Rational functions: $\dfrac{p(x)}{q(x)}$

where p and q are polynomials are continuous at all points a where $q(a) \neq 0$.

3. $\cosh x = \dfrac{e^x + e^{-x}}{2}$ $\quad \sinh x = \dfrac{e^x - e^{-x}}{2}$

are continuous.

Theorem If g is continuous at a and f is continuous at $g(a)$ then $f \circ g$ is continuous at a.

In particular

$$\lim_{x \to a} (f \circ g)(x) = \lim_{x \to a} f(g(x))$$

$$= f\left(\lim_{x \to a} g(x)\right).$$

(substitution law for limits)

<u>Example</u>

Let $f(x) = \sin x$, $g(x) = 2x^2+1$.

Then g is continuous and f is continuous so e.g.

$$\lim_{x \to 5} \sin(2x^2+1) = \sin\left(\lim_{x \to 5} (2x^2+1)\right)$$

$$= \sin 51.$$

<u>Exercise</u> Prove that for all $a>0$

$$\lim_{x \to a} \sqrt{x} = \sqrt{a}.$$

Hence $f(x) = \sqrt{x}$ is continuous on $(0, \infty)$.

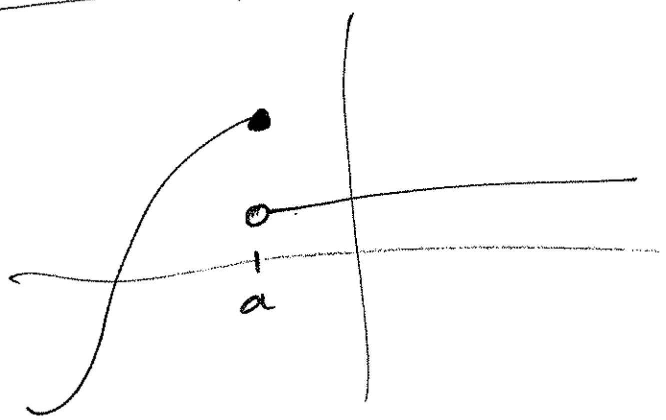
Left and right continuity

The function f is continuous from the left at a if
$$\lim_{x \to a^-} f(x) = f(a)$$

and continuous from the right at a if
$$\lim_{x \to a^+} f(x) = f(a).$$

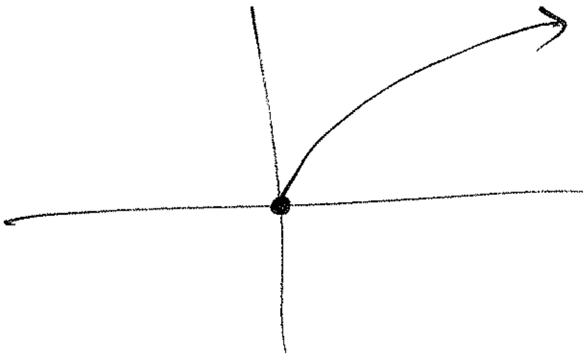
Examples

1.

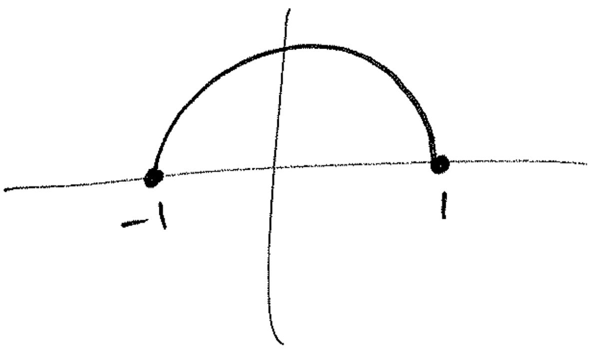


f is continuous from the left at a but not continuous from the right at a, since even though $\lim_{x \to a^+} f(x)$ exists, it does not equal $f(a)$.

2. $f(x) = \sqrt{x}$ is continuous from the right at 0 since (exercise) $\lim_{x \to 0^+} \sqrt{x} = 0 = f(0)$.



2. $f(x) = \sqrt{1-x^2}$ is continuous from the right at $x = -1$ and continuous from the left at $x = 1$.



Continuity on intervals $a, b \in \mathbb{R}$

1. f is continuous on the open interval (a,b) if f is continuous at each point in (a,b).
2. f is continuous on the closed and bounded interval $[a,b]$ if (here $a<b$ real)

f is continuous on (a,b),
f is continuous from the right at
a and f is continuous from
the left at b.

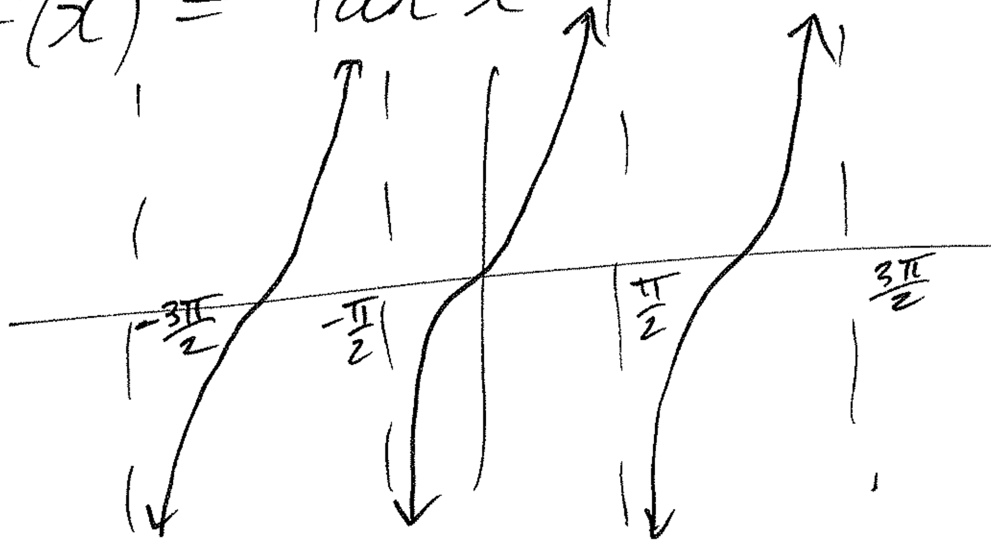
3. Similarly for continuity on
$[a,b)$ and $(a,b]$.

4. f is continuous on (a,∞)
if f is continuous at each point
in (a,∞) i.e. at each point $>a$.
Similarly for $(-\infty,b)$.

5. f is continuous on $[a,\infty)$
if continuous on (a,∞)
and continuous from the
right at a.
Similarly for $(-\infty,b]$.

Examples

1. $f(x) = \tan x$



is continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

2. $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

3. $f(x) = \sqrt{1-x^2}$ is continuous on $[-1, 1]$.

Upper and lower bounds on sets of real numbers

(Note: ∞ and $-\infty$ are not real numbers)

A set S of real numbers is <u>bounded above</u> if there is an $\alpha \in \mathbb{R}$ so that for all $a \in S$,

$$a \leq \alpha.$$

α alpha

Such an α is called an <u>upper bound</u> for the set S.

A number $\alpha \in \mathbb{R}$ is a <u>least upper bound</u> (l.u.b.) for the set S if α is an upper bound for S <u>and</u> if β is any upper bound for S,

beta

$$\alpha \leq \beta.$$

i.e. α is less than or equal to any other upper bound, it is the best upper bound.

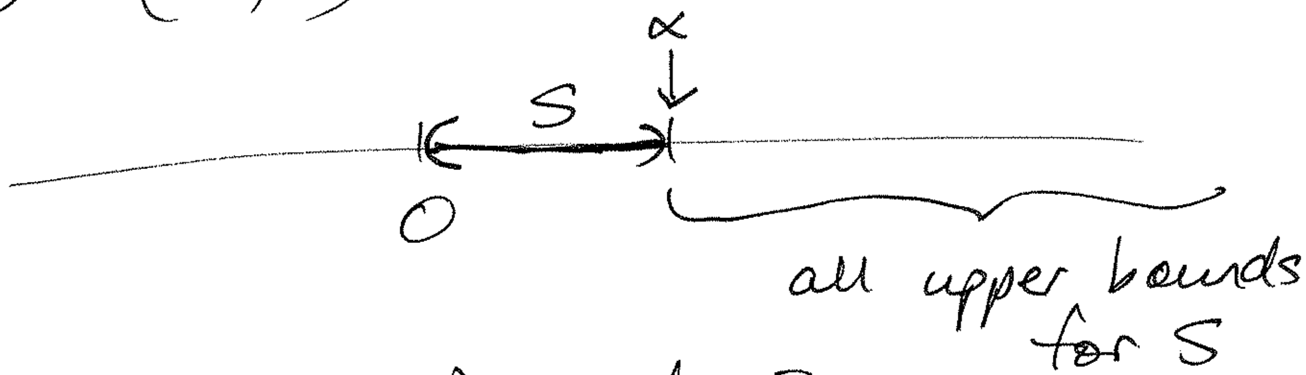
Examples

1. $S = [0,1]$

Any $\alpha \geq 1$ is an upper bound for S.

The least upper bound is $\alpha = 1$.

2. $S = (0,1)$ has l.u.b. $\alpha = 1$.



Least Upper Bound Property

If S is any nonempty set of real numbers which is bounded above, then S has a least upper bound, and this least upper bound is unique.
(Similarly for lower bounds.)