Week 5 Lecture 2

A set S of real numbers is **bounded above** if there is $\alpha \in \mathbb{R}$ so that $\alpha \geq a$ for all $a \in S$.

An **upper bound** for S is an $\alpha \in \mathbb{R}$ so that $\alpha \geq a$ for all $a \in S$.

So S is bounded above $\iff$ S has an upper bound.
if and only if

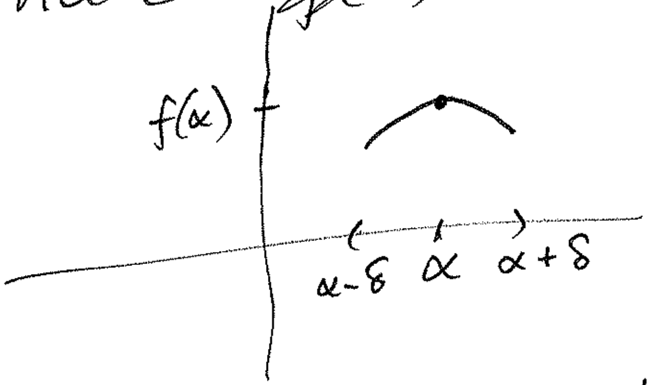
A **least upper bound** for S is an $\alpha \in \mathbb{R}$ which is an upper bound for S, such that if β is any other upper bound for S, then

$$\alpha \leq \beta.$$

Least Upper Bound Property If S is nonempty and bounded above, then S has a unique least upper bound.

<u>Lemma</u> Suppose f is continuous at α. (i.e. $\lim_{x \to \alpha} f(x) = f(\alpha)$).

1. If $f(\alpha) > 0$ then there is $\delta > 0$ so that whenever $|x - \alpha| < \delta$, we have ~~$f(x) > 0$~~ $f(x) > 0$.



2. If $f(\alpha) < 0$ then there is $\delta > 0$ so that whenever $|x - \alpha| < \delta$, we have $f(x) < 0$.

<u>Proof</u> 1. Let $\varepsilon = \frac{f(\alpha)}{2} > 0$.

Then since $\lim_{x \to \alpha} f(x) = f(\alpha)$, there is a $\delta > 0$ such that whenever $0 < |x - \alpha| < \delta$, we have $|f(x) - f(\alpha)| < \varepsilon$.

Now $|f(x) - f(\alpha)| < \varepsilon = \frac{f(\alpha)}{2}$

means exactly that

$$f(\alpha) - \frac{f(\alpha)}{2} < f(x) < f(\alpha) + \frac{f(\alpha)}{2}$$

so in particular for $0 < |x-\alpha| < \delta$,

we have $\frac{f(\alpha)}{2} < f(x)$.

Thus for $|x-\alpha| < \delta$, since $f(\alpha) > 0$,

$$f(x) > 0.$$

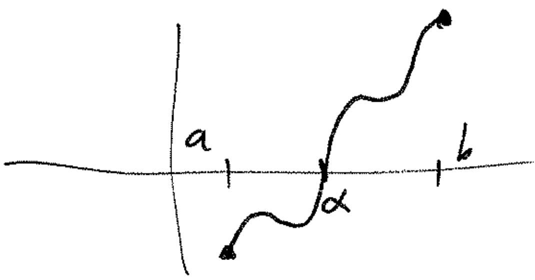
2. Let $\varepsilon = \frac{-f(\alpha)}{2} > 0$.

<u>Theorem</u> Suppose f is continuous on a closed and bounded interval $[a,b]$. If either

1. $f(a) < 0$ and $f(b) > 0$

or 2. $f(a) > 0$ and $f(b) < 0$

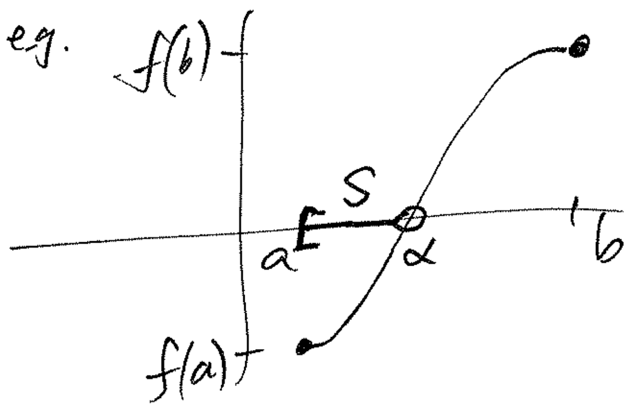
then there is an $\alpha \in (a,b)$ so that $f(\alpha) = 0$.

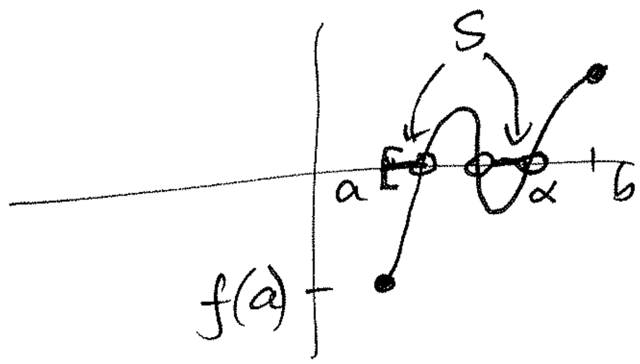


Proof 1. $f(a) < 0 < f(b)$.

Let $S = \{ x \in [a, b] \mid f(x) < 0 \}$

e.g.





The set S is nonempty since $a \in S$.

The set S is bounded above since b is an upper bound for S.

So by the Least Upper Bound Property, S has a unique least upper bound α.

Claim $f(\alpha) = 0$.

Proof of Claim by contradiction.

Assume ~~$f(\alpha) > 0$~~ $f(\alpha) < 0$.

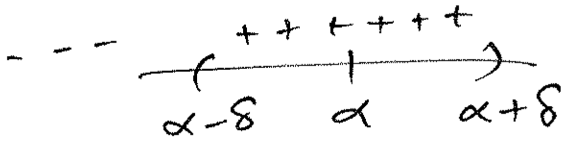
By the Lemma, there is a $\delta > 0$ so that if x is in $(\alpha - \delta, \alpha + \delta)$, $f(x) < 0$.

So in particular all x in $(\alpha, \alpha+\delta)$ are in S. $|x-\alpha|<\delta$

But this contradicts α an upper bound for S. So $f(\alpha) \geq 0$.

Assume $f(\alpha) > 0$. Then $\alpha \neq a$.

By the Lemma, there is a $\delta > 0$ so that for all x in $(\alpha-\delta, \alpha+\delta)$ we have $f(x) > 0$.



This contradicts α being the <u>least</u> upper bound for S.

Therefore $f(\alpha) = 0$, proving the Claim. //

Continuing proof of Theorem:

we have $f(\alpha) = 0$, $f(a) \neq 0$, $f(b) \neq 0$

so $\alpha \in (a, b)$.

2. $f(a) > 0 > f(b)$. Put $g(x) = -f(x)$.

Then ~~$g(a) > 0 > g(b)$~~

$$g(a) < 0 < g(b)$$

and apply part 1.

Intermediate Value Theorem

If f is continuous on $[a,b]$

and either 1. $f(a) < k < f(b)$

or 2. $f(a) > k > f(b)$

then there is some $\alpha \in (a,b)$ so that $f(\alpha) = k$. (previous Theorem was the case $k=0$)

Proof Let $g(x) = f(x) - k$.

Then g is continuous on $[a,b]$

~~so by previous Theorem g~~

and either 1. $f(a) - k < 0 < f(b) - k$

so $g(a) < 0 < g(b)$

or 2. $g(a) > 0 > g(b)$

so by prev. Thm there is an $\alpha \in (a,b)$ ~~such that g~~

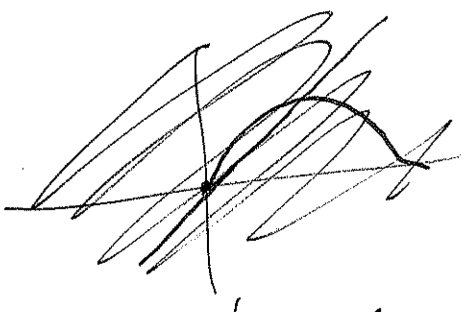
So that $g(\alpha) = 0$.
Thus $f(\alpha) - k = 0$ so $f(\alpha) = k$.

Examples 1. Show that there is a positive real number x such that $x^2 = 2$.

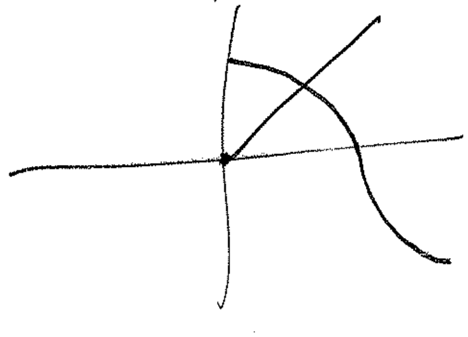
Let $f(x) = x^2$. Now $f(1) = 1$, ($a = 1$)
$f(2) = 4$ ($b = 2$) and $1 < 2 < 4$
so by IVT (Intermediate Value Theorem)

there is an $x \in (1, 2)$ so that
$f(x) = 2$. That is, $x^2 = 2$.

2. Show that $x = \cos x$ has a solution in $\left(0, \frac{\pi}{2}\right)$.



Let $f(x) = x - \cos x$.

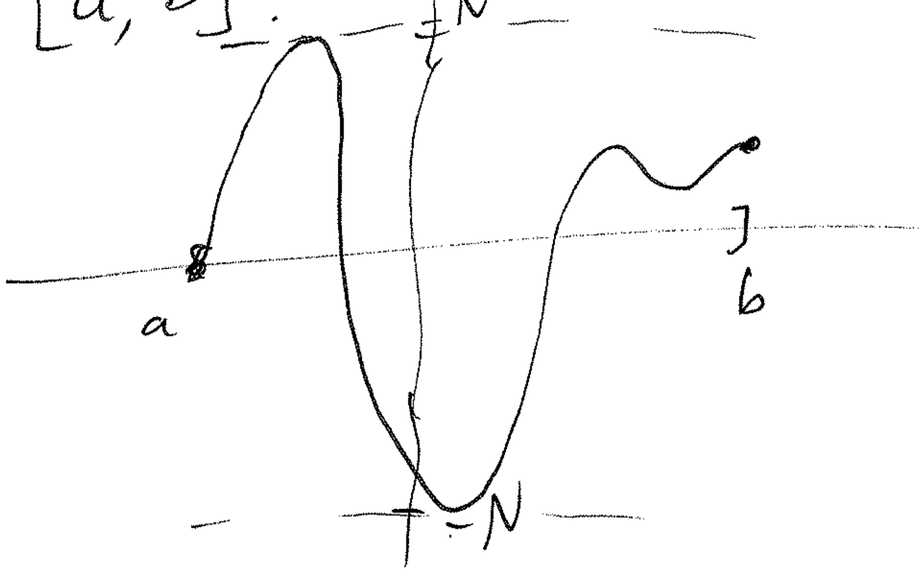


Then f is continuous on $\left[0, \frac{\pi}{2}\right]$ and
$f(0) = -1$, $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

so by IVT there is $\alpha \in (0, \frac{\pi}{2})$ such that $f(\alpha) = 0$ thus

$$\alpha = \cos \alpha .$$

<u>Theorem</u> If f is continuous on $[a, b]$ then there is an $N \geq 0$ so that $|f(x)| \leq N$ for all x in $[a, b]$.



(A continuous function on a closed and bounded interval is bounded.)

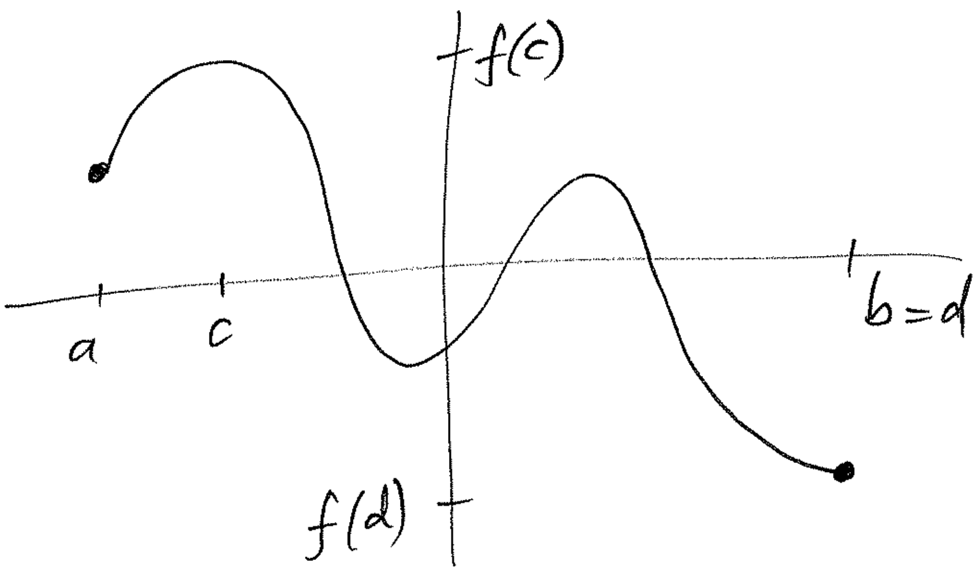
<u>Example</u> $f(x) = \frac{1}{x}$ is bounded on $[0.00001, 5]$ but not on $(0, 5]$.

Exercise Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 0$, then f is bounded on $\mathbb{R}$ (i.e. there is an $N \geq 0$ so that for all $x \in \mathbb{R}$, $|f(x)| \leq N$.)

Extreme Value Theorem

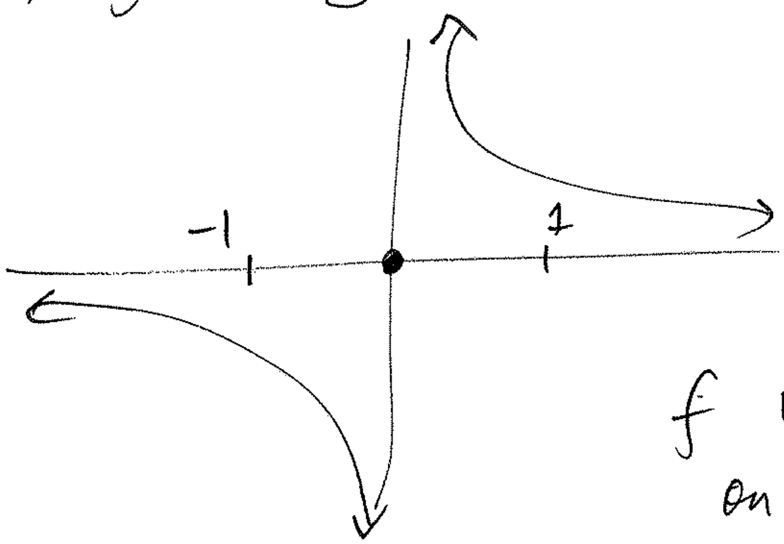
If f is continuous on $[a, b]$, then

1. there is a $c \in [a,b]$ so that $f(c)$ is the absolute maximum of f on $[a,b]$ i.e. $f(c) \geq f(x)$ for all $x \in [a,b]$.
2. there is a $d \in [a,b]$ so that $f(d)$ is the absolute minimum of f on $[a,b]$ i.e. $f(d) \leq f(x)$ for all $x \in [a,b]$



(A continuous function on a closed and bounded interval achieves its maximum and minimum values.)

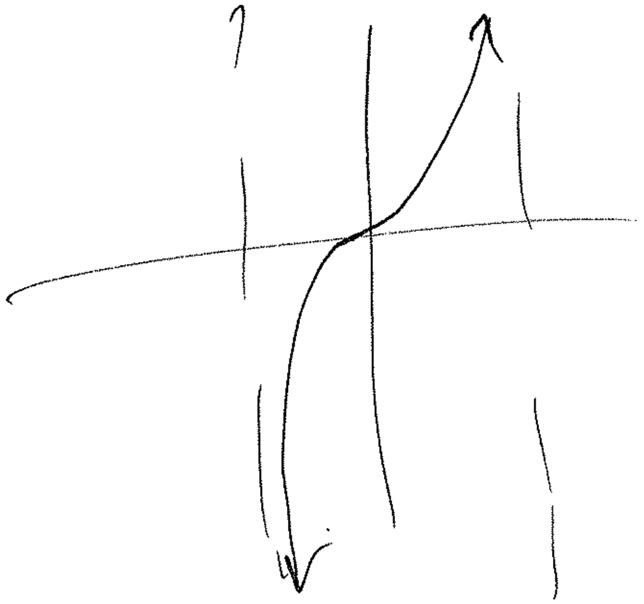
<u>Note</u> 1. Must have f continuous on $[a, b]$ e.g. $f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$



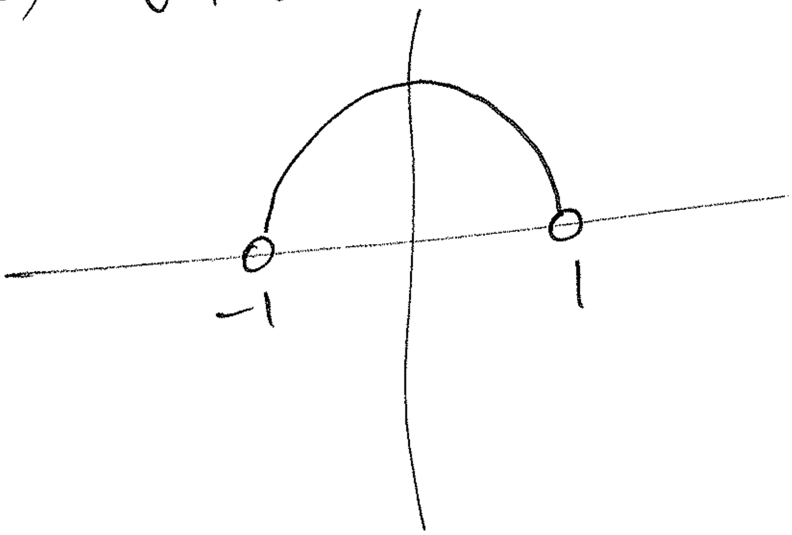
f is not bounded on $[-1, 1]$, and does not have absolute max. or min. on $[-1, 1]$

2. Must be closed interval
i.e. including endpoints

e.g. $f(x) = \tan x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$f(x) = \sqrt{1-x^2}$ on $(-1, 1)$



has absolute maximum $f(0) = 1$
but no absolute minimum on $(-1, 1)$.

These theorems boil down to:
image of ~~[?]~~ $[a, b]$ under a continuous function is another closed + bounded interval.