Week 6 Lecture 1

Next week in tutorials: Quiz.
on Weeks 1-~~6~~5 in lectures
Weeks 2-6 in tutorials.

Closed, open and bounded sets

A set of real numbers S is 1. open if for every $x \in S$ there is an $\varepsilon > 0$ so that the interval $(x-\varepsilon, x+\varepsilon)$ is contained in S.

2. closed if its complement
$$S^c = \{ x \in \mathbb{R} \mid x \notin S \}$$
is open

3. bounded if it is bounded above and bounded below i.e. there exist $\alpha, \beta \in \mathbb{R}$ so that for all $x \in S$, $\alpha \leq x \leq \beta$.

The following intervals are ~~open~~

open: (a,b), (a,∞), $(-\infty,a)$, $\underset{\mathbb{R}}{\underset{\shortparallel}{(-\infty,\infty)}}$

closed: $[a,b]$, $(-\infty, a]$, $[a,\infty)$, $\underset{\mathbb{R}}{\underset{\shortparallel}{(-\infty,\infty)}}$

(the empty set is open)

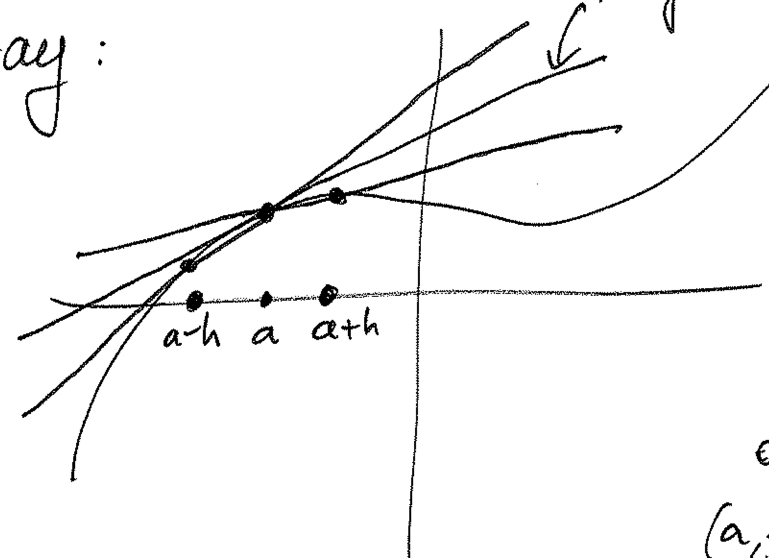
bounded: $[a,b]$, $\underbrace{[a,b),\ (a,b]}_{\text{neither open nor closed}}$

Thus the only ~~set~~ intervals which are both ~~op~~ closed and bounded are those of the form $[a,b]$.

Differentiation

See Thurston essay for many ways of thinking about the derivative.

One way:



As $h \to 0$ (h positive or negative), slope of lines joining $(a,f(a))$ to $(a \pm h, f(a \pm h))$

approaches the slope of the tangent line to the graph of $f(x)$ at $(a, f(a))$. ③

<u>Definition</u> A function f is <u>differentiable</u> at a point a in its domain if the limit

$$\lim_{h \to 0} \underbrace{\frac{f(a+h) - f(a)}{h}}_{\text{slope of line joining } (a, f(a)) \text{ to } (a+h, f(a+h))}$$

exists. If this limit exists, its value is called the <u>derivative</u> of $f(x)$ at a, denoted $f'(a)$.

<u>Example</u>

1. ~~The~~ For any constant k, the function $f(x) = k$ is differentiable at any $a \in \mathbb{R}$ since

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{k - k}{h}$$

$$= \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0.$$

2. The function $f(x) = x$ is differentiable at all $a \in \mathbb{R}$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{a+h-a}{h}$$

$$= \lim_{h \to 0} \frac{h}{h}$$

$$= \lim_{h \to 0} 1$$

$$= 1.$$

<u>Theorem</u> If f is differentiable at a then f is continuous at a.

<u>Proof</u> Want to show $\lim_{x \to a} f(x) = f(a)$.

Will use these 2 facts about limits:

1. $\lim_{x \to a} f(x) = f(a) \iff \lim_{x \to a} (f(x) - f(a)) = 0$
2. $\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$. (put $x = a+h$)

Now

$$\lim_{x \to a} (f(x) - f(a)) = \lim_{h \to 0} (f(a+h) - f(a))$$

$$= \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \cdot h \right)$$

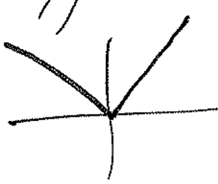
$$= \left(\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \right) \left(\lim_{h \to 0} h \right)$$

since both of these limits exist; the first one because f is differentiable at a

$$= (f'(a))(0)$$

$$= 0$$

thus $\lim_{x \to a} f(x) = f(a)$ as required.

Note the converse is not true 

e.g. $f(x) = |x|$ is continuous on $\mathbb{R}$, but not differentiable at 0 since

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1$$

Thus the left- and right-hand limits exist but they are not the same so $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ does not exist.

Definition f is left-differentiable at a if $\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$, the left-hand derivative exists, similarly f is right-differentiable at a if the right-hand derivative $\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$

exists.

So f is differentiable at a

$\iff$ both left- and right-hand derivatives exist and are equal.

~~Differentiability on intervals~~

~~f is differentiable at each point in an open interval~~

Differentiability on intervals

Let I be an open interval. Then f is differentiable on I if it is differentiable at each point in I.

If f is differentiable on I then we define a function f' on I, called the derivative of f by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

If f' is differentiable on I then we can define the second derivative by

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

third derivative (if f'' is differentiable)

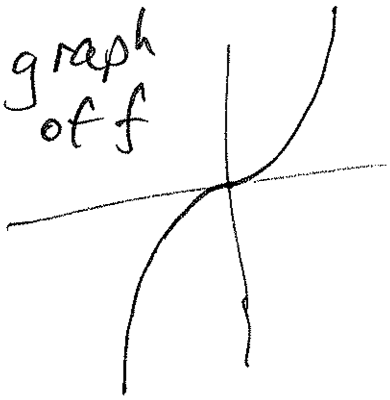
$$f^{(3)}(x) = f'''(x) = \lim_{h \to 0} \frac{f''(x+h) - f''(x)}{h}$$

etc.

<u>Note</u> For any positive integer n, there is a function f so that the n^{th} derivative $f^{(n)}$ exists, but $f^{(n+1)}$ does not i.e. $f^{(n)}$ is not differentiable.

e.g. $n=1$

$$f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$$

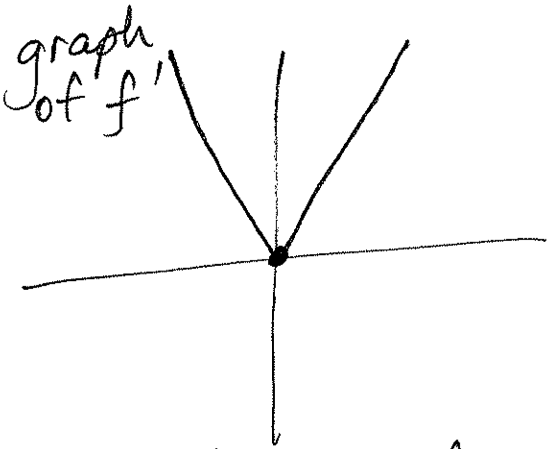


is differentiable at 0

$$f'(0) = 0$$

but

$$f'(x) = \begin{cases} 2x & x > 0 \\ 0 & x = 0 \\ -2x & x < 0 \end{cases}$$



is not differentiable at 0.

Leibniz notation

$$f' \longrightarrow \frac{df}{dx} \text{ or } \frac{d}{dx}f$$

$$f'(a) \longrightarrow \frac{df}{dx}(a) \text{ or } \left.\frac{df}{dx}\right|_{x=a}$$

$$f'' \longrightarrow \frac{d^2 f}{dx^2} \text{ or } \frac{d}{dx}\left(\frac{df}{dx}\right) \text{ or } \frac{d^2}{dx^2} f$$

$$f^{(n)} \longrightarrow \frac{d^n f}{dx^n} \qquad \text{etc.}$$