Week 6 Lecture 2

f is <u>differentiable at a</u> if

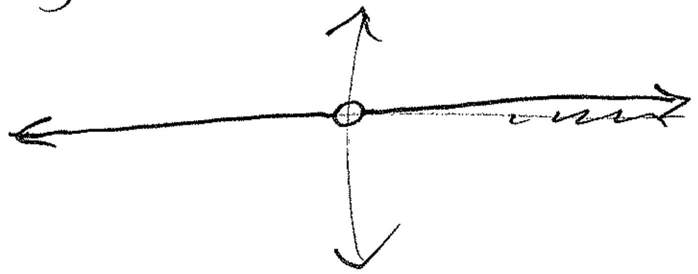
$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. Denote limit by $f'(a)$.

e.g. $f(x) = k$ constant function is differentiable at all $a \in \mathbb{R}$.

Let $g(x) = \frac{0}{x}$ be defined for all $x \neq 0$.

Then $g(x) = 0$ for all $x \neq 0$.



graph of g

so $\lim_{x \to 0} g(x) = 0$

So

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{k - k}{h}$$

$$= \lim_{h \to 0} \frac{0}{h}$$

$$= \lim_{h \to 0} g(h) = 0.$$

for all a.

Thus $f'(a) = 0$

Properties of differentiation

Suppose f and g are differentiable on an open interval I.

Then $f+g$, $f-g$, fg, $\frac{f}{g}$ (for $g(x) \neq 0$ on I) and kf (k constant) are all differentiable on I and

1. $(f+g)' = f' + g'$ or $\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$
2. $(f-g)' = f' - g'$ or $\frac{d}{dx}(f-g) = \frac{df}{dx} - \frac{dg}{dx}$
3. Product rule
   $(fg)' = f'g + g'f$ or $\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$
4. Quotient rule
   $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$ or $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{df}{dx}g - f\frac{dg}{dx}}{g^2}$
5. $(kf)' = kf'$ or $\frac{d}{dx}(kf) = k\frac{df}{dx}$.

Proofs All come from definition of the derivative as a limit, and limit laws.

1. ~~lim~~

$$(f+g)'(x) = \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h}$$

$$= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right]$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

since f and g are differentiable

$$= f'(x) + g'(x).$$

3. $$(fg)'(x) = \lim_{h\to 0} \frac{(fg)(x+h) - (fg)(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h} \right]$$

$$= \left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right)\left(\lim_{h\to 0} g(x+h)\right) + \left(\lim_{h\to 0} f(x)\right)\left(\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right)$$

since all of these limits exist

$$= f'(x)\,g(x) + f(x)\,g'(x).$$

Chain Rule

If f and g are differentiable then

$$\underbrace{(f\circ g)}_{\substack{\text{composite}\\ \text{function}}}{}'(x) = f'(g(x))\,g'(x)$$

$(f\circ g)(x) = f(g(x))$

i.e. the derivative of $f\circ g$ at x equals the derivative of f at $g(x)$ multiplied by the derivative of g at x.

Proof see Spivak.

Examples

1. For all positive integers n

$$\frac{d}{dx}\left(x^n\right) = n x^{n-1}$$

proof by induction.

For all positive integers n

$$\frac{d}{dx}\left(x^{-n}\right) = (-n)\, x^{-n-1}.$$

e.g. $$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}.$$

2. Polynomials are differentiable.

e.g. $\frac{d}{dx}(3x^3 + 4x^2 - 17x + 5)$

$= 9x^2 + 8x - 17$

3. Rational functions are differentiable, at all points a where $q(a) \neq 0$.

$\frac{p(x)}{q(x)}$ p, q polynomials

4. e^x, $\ln x$, $\sin x$, $\cos x$ are all differentiable with derivatives

e^x, $\frac{1}{x}$, $\cos x$, $-\sin x$

5. $\cosh x$ and $\sinh x$ are differentiable

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{1}{2}(e^x - e^{-x})$$

$$= \sinh x$$

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^{x}-e^{-x}}{2}\right)$$

$$= \frac{e^{x}+e^{-x}}{2}$$

$$= \cosh x.$$

5. $F(x) = \sin(x^2 e^{3x})$

then $F = f \circ g$

where $f(x) = \sin x$ $\quad f'(x) = \cos x$

$g(x) = x^2 e^{3x}$

so by Chain Rule

$$F'(x) = f'(g(x))\, g'(x)$$

$$= \cos(x^2 e^{3x})\, g'(x)$$

To compute $g'(x)$ write

$$g(x) = h(x)k(x)$$

where $h(x) = x^2$ so $h'(x) = 2x$

$k(x) = e^{3x}$ so

$k'(x) = 3e^{3x}$

so by Product Rule

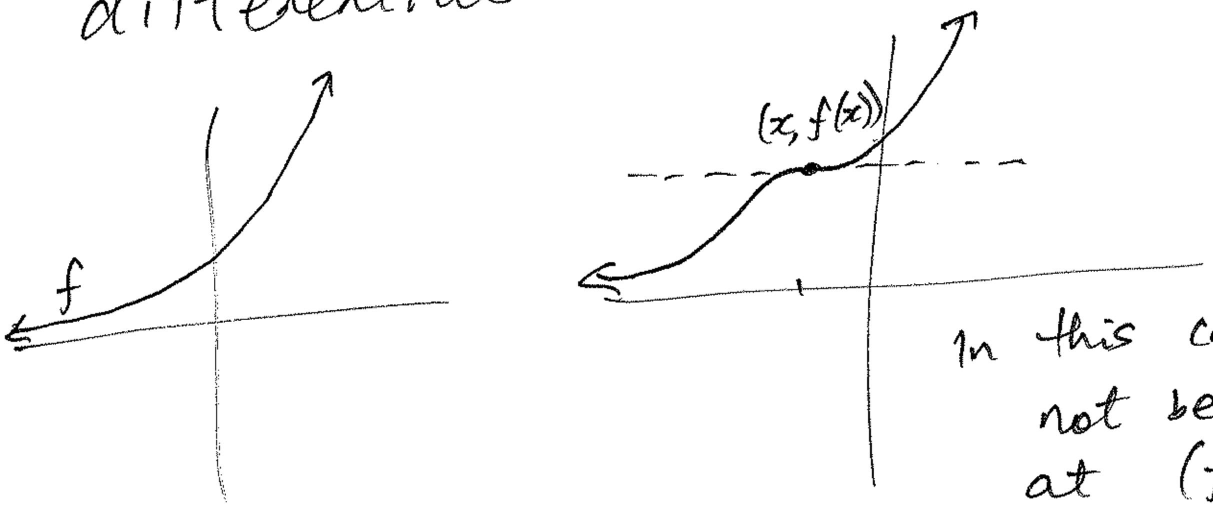
$$g'(x) = 2xe^{3x} + 3x^2e^{3x}$$

so

$$F'(x) = \cos(x^2 e^{3x})\left[2xe^{3x} + 3x^2 e^{3x}\right]$$

Derivatives of inverse functions

Suppose f is invertible and f is differentiable at x with $f'(x) \neq 0$.



In this case, f^{-1} will not be differentiable at $(f(x), x)$ since vertical tangent line at $(f(x), x)$.

Then f^{-1} is differentiable at $f(x)$

and

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

Proof Write $g = f^{-1}$. Then

$$x = g(f(x)).$$

with respect to

Differentiate both sides w.r.t. x.

$$\frac{d}{dx}\left(x\right) = \frac{d}{dx}\left(g(f(x))\right)$$

$$1 = g'(f(x))f'(x)$$

by Chain Rule

so since $f'(x) \neq 0$

$$g'(f(x)) = \frac{1}{f'(x)}$$

but $g = f^{-1}$ so

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

<u>Example</u> Let $f(x) = \sinh x$, so $f'(x) = \cosh x$

By theorem,

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

so

$$(\sinh^{-1})'(\sinh x) = \frac{1}{\cosh x}.$$

To get a formula for $(\sinh^{-1})'$, put $y = \sinh x$. Now

$$\cosh^2 x - \sinh^2 x = 1 \quad \text{so}$$

so

$$\cosh x = \sqrt{1 + \sinh^2 x}$$

since $\cosh x \geq 1$ for all x

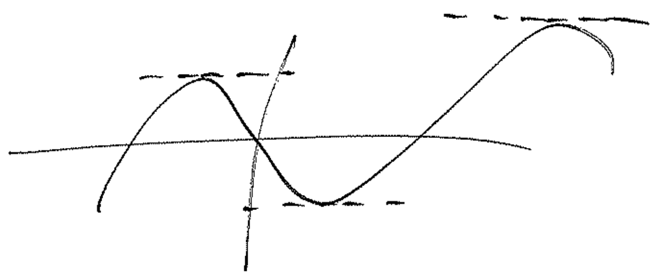
so

$$(\sinh^{-1})'(y) = \frac{1}{\sqrt{1+y^2}}$$

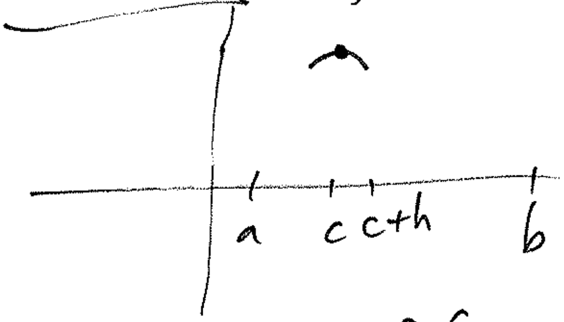
since $y = \sinh x$.

Maxima and minima

Theorem If f is differentiable on (a,b) and f has a maximum or minimum at $c \in (a,b)$ then $f'(c) = 0$.



Proof f has max. at c.

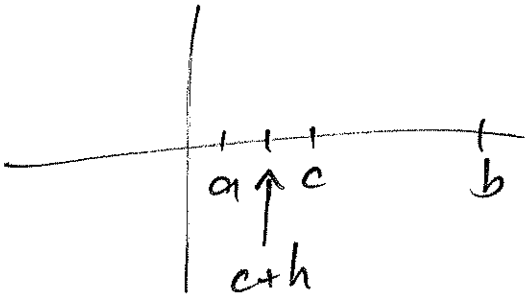


Choose h so that $c < c+h < b$.

Then $\dfrac{f(c+h) - f(c)}{h} \leq 0$ since $f(c+h) \leq f(c)$ and $h > 0$.

So $\lim\limits_{h\to 0^+} \dfrac{f(c+h)-f(c)}{h} \leq 0$.

Choose h so that $a < c+h < c$.



Then

$$\frac{f(c+h)-f(c)}{h} \geq 0$$

since $f(c+h) \leq f(c)$ and $h<0$.

so $\lim\limits_{h\to 0^-} \dfrac{f(c+h)-f(c)}{h} \geq 0$.

Now f is differentiable at c

so $\lim\limits_{h\to 0} \dfrac{f(c+h)-f(c)}{h}$ exists

and equals $\lim\limits_{h\to 0^-} \dfrac{f(c+h)-f(c)}{h}$ and $\lim\limits_{h\to 0^+} \dfrac{f(c+h)-f(c)}{h}$.

Hence $\cancel{\lim\limits_{h\to 0^+}}\ f'(c) = 0$.

Similarly for min. at c.

(or consider $g(x) = -f(x)$.)

Note Converse is not true

e.g. $f(x) = x^3$

$f'(0) = 0$ but f does not have max. or min. at 0.

Finding max. and min. values

A critical point for f is a value x so that either $f'(x) = 0$ or f' does not exist.

e.g. if $f(x) = |x|$ then 0 is critical point.

If f is continuous on $[a,b]$, then the maximum and min. values of f occur either at critical points in (a,b) or at endpoints $x=a$ or $x=b$.

Example Find max and min values of $f(x) = x^3 - x$ on $[-1,2]$.

f is diff. on $(-1,2)$

critical points are solutions to

$f'(x)=0$ i.e. $3x^2-1=0$

i.e. $x = \pm\frac{1}{\sqrt{3}}$.

Check function values at critical points and endpoints

$f(-1)=0$ $f\left(-\frac{1}{\sqrt{3}}\right)=\frac{2}{3\sqrt{3}}$ $f\left(\frac{1}{\sqrt{3}}\right)=\frac{-2}{3\sqrt{3}}$

and $f(2)=6$

so max. value is 6

min. " " $\frac{-2}{3\sqrt{3}}$.