Week 7

Assignment 1 (the only assignment) is now available. Due Thurs 16 May. After today's lecture you can do all questions except 4(b).

Today: • 3 theorems about differentiable functions

• L'Hôpital's rule

Rolle's Theorem

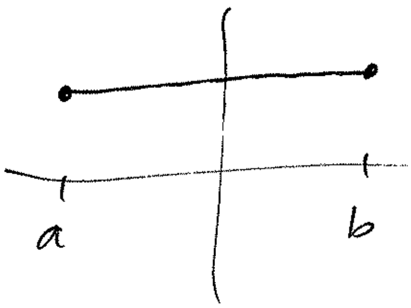
Suppose f is continuous on $[a,b]$, differentiable on (a,b) and $f(a)=f(b)$. Then there is a c in (a,b) so that

$$f'(c)=0.$$

Proof By Extreme Value Theorem, f attains its minimum and its maximum on $[a,b]$.

Case 1 f attains either its minimum or its maximum in (a,b).
Then by a theorem in previous lecture, there is a c in (a,b) so that $f'(c)=0$.

Case 2 both minimum and maximum occur at endpoints.



Then as $f(a)=f(b)$, f is a constant function on $[a,b]$.

Thus $f'(c)=0$ for all c in (a,b) in Case 2.

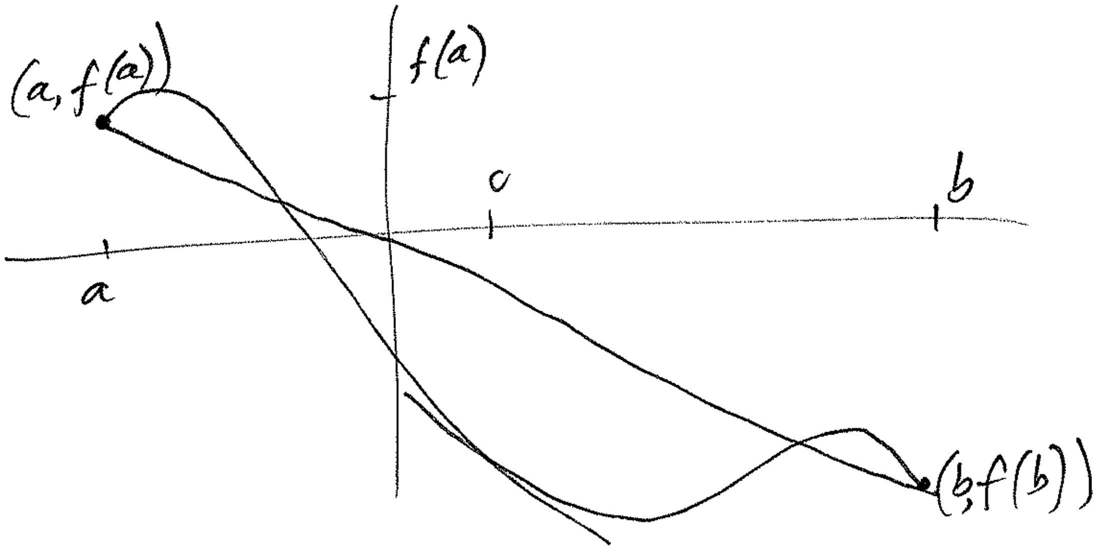
Thus in both cases, there is a $c \in (a,b)$ so that $f'(c)=0$.

Mean Value Theorem

Suppose f is continuous on $[a,b]$ and differentiable on (a,b). Then there exists some $c \in (a,b)$ so that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Rolle's Thm: special case $f(a)=f(b)$



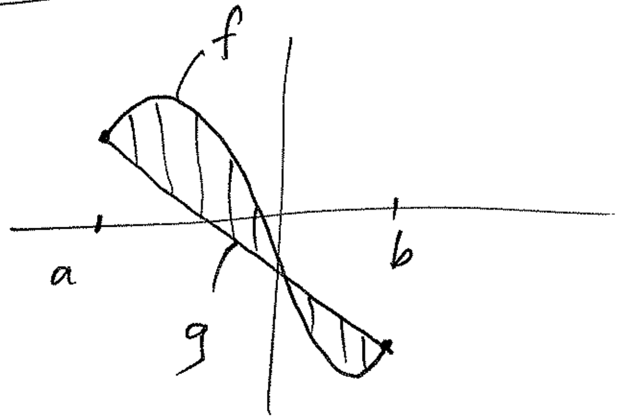
Line has equation

$$y - f(a) = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{slope}}(x-a)$$

MVT says there is a c in (a,b) so that slope of tangent line at c equals slope of line joining $(a, f(a))$ to $(b, f(b))$.

Another point of view: $\frac{f(b)-f(a)}{b-a}$ is average (or mean) value of f' over $[a,b]$ e.g. if f is position, at time x average value of f' is average ~~speed~~ velocity over $[a,b]$. MVT says that instantaneous velocity equals the average velocity at some time c.

Proof of MVT



Define

$$g(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$$

equation of line

$$h(x) = f(x) - g(x)$$

Then h is continuous on $[a,b]$ and differentiable on (a,b) and

$$h(a) = f(a) - g(a)$$
$$= f(a) - \frac{f(b)-f(a)}{b-a}(a-a) - f(a)$$
$$= 0$$
$$h(b) = f(b) - g(b)$$
$$= f(b) - \frac{f(b)-f(a)}{b-a}(b-a) - f(a)$$
$$= f(b) - f(b) + f(a) - f(a)$$
$$= 0.$$

So we may apply Rolle's Theorem to h. Thus there is a $c \in (a,b)$ so that $h'(c) = 0$. Now

$$0 = h'(c) = f'(c) - g'(c)$$

So

$$f'(c) = g'(c)$$
$$= \frac{f(b) - f(a)}{b-a}.$$

since $g(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$

so $g'(x) = \frac{f(b)-f(a)}{b-a}$.

Important corollaries of MVT

In each statement, I is an open interval.

1. If f is defined on I and $f'(x)=0$ for all $x \in I$ then f is a constant function on I.

Proof Choose $a, b \in I$ with $a<b$.
Then by the MVT, there is a $c \in (a,b)$ so that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Now $f'(c)=0$ so $f(b)-f(a)=0$ thus $f(b)=f(a)$. This holds for all $a,b \in I$, so f is constant

2. If f and g are defined on I and $f'(x) = g'(x)$ on I then there is a constant k so that for all $x \in I$, $f(x) = g(x) + k$. 6

<u>Proof</u>. Let $h(x) = f(x) - g(x)$. Then

$$h'(x) = f'(x) - g'(x) = 0$$

for all $x \in I$, so by previous result h is constant function $h(x) = k$ on I. Thus $f(x) = g(x) + k$ on I.

3. If $f'(x) > 0$ for all $x \in I$ then f is increasing on I.
(Similarly $f'(x) < 0$ on I then f is decreasing.)

<u>Proof</u> We need to show that if $x_1, x_2 \in I$ with $x_1 < x_2$ then $f(x_1) < f(x_2)$.
By MVT on $[x_1, x_2]$ there is a ~~cel~~ c with $x_1 < c < x_2$ so that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Now $f'(c) > 0$ so $f(x_2) - f(x_1) > 0$.
Thus $f(x_1) < f(x_2)$ as required.

Cauchy Mean Value Theorem

If f and g ~~and~~ are continuous on $[a,b]$ and differentiable on (a,b) then there is some $c \in (a,b)$ so that

$$(f(b) - f(a))\,g'(c) = (g(b) - g(a))\,f'(c).$$

(MVT is special case $g(x) = x$
Rolle's Thm is special case $g(x) = x$ and $f(a) = f(b)$.)

<u>Proof</u> Apply Rolle's Thm to

$$h(x) = f(x)\big(g(b) - g(a)\big) - g(x)\big(f(b) - f(a)\big)$$

L'Hôpital's Rule

l'Hôpital is contraction of
le Hôpital.

H is silent.

One form of l'Hôpital's rule is:

Suppose f and g are differentiable at all points x in an open interval I containing a, except possibly at a.

If
1. $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$
2. $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)} = l$
3. $g'(x) \neq 0$ for all $x \in I$, $x \neq a$

Then
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = l.$$

Proof Uses Cauchy MVT.

Variations on l'Hôpital's rule

Instead of $x \to a$, can have $x \to a^+$,

$x \to a^-$, $x \to \infty$, $x \to -\infty$.

Instead of $f(x) \to 0$ and $g(x) \to 0$

can have $f(x) \to \infty$ and $g(x) \to \infty$

or $f(x) \to -\infty$ and $g(x) \to -\infty$

Instead of limit $l \in \mathbb{R}$ can have

$$\frac{f'(x)}{g'(x)} \to \infty \quad \text{or} \quad \frac{f'(x)}{g'(x)} \to -\infty .$$

Examples

1. $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$ of form "$\dfrac{0}{0}$"

$$\lim_{x \to 0} \frac{\cos x}{1} = \lim_{x \to 0} \cos x = 1 \quad (= l)$$

So by L'Hôpital's rule

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 .$$

2. $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{\sin x} = \lim\limits_{x \to 0} \dfrac{\sin x}{\cos x} = \dfrac{0}{1} = 0.$

3. $\lim\limits_{x \to \infty} \dfrac{\ln x}{x}$ of form "$\dfrac{\infty}{\infty}$"

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

4. $$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

2 applications of L'Hôpital's rule

Exercise : for any polynomial p

$$\lim_{x\to\infty} \frac{e^x}{p(x)} = \infty .$$

5. $\lim_{x\to 0} (1+kx)^{\frac{1}{x}}$ here k is a constant

In order to apply L'Hôpital's rule, ~~I~~ apply the function $\ln$.

Put $y = (1+kx)^{1/x}$

then $\ln y = \frac{1}{x} \ln(1+kx)$

$$= \frac{\ln(1+kx)}{x}$$

By L'Hôpital's rule

$$\lim_{x\to 0} \frac{\ln(1+kx)}{x} = \lim_{x\to 0} \frac{\frac{k}{1+kx}}{1}$$

$$= \lim_{x\to 0} \frac{k}{1+kx} = k.$$

So we have

$$\lim_{x \to 0} (\ln y) = k.$$

Apply exp to both sides

$$e^{\lim_{x \to 0} (\ln y)} = e^k$$

$$\lim_{x \to 0} \left(e^{\ln y}\right) = e^k$$

$$\lim_{x \to 0} y = e^k$$

$$\lim_{x \to 0} (1+kx)^{1/x} = e^k .$$

In particular if $k=1$

$$\lim_{x \to 0} (1+x)^{1/x} = e .$$