Week 8 Lecture 1

L'Hôpital's rule: nothing to do with quotient rule for differentiation.

Informally, l'Hôpital's rule says:

If $\lim_{x \to \cdot} f(x) = \lim_{x \to \cdot} g(x) = 0, \infty, -\infty$

and $\lim_{x \to \cdot} \frac{f'(x)}{g'(x)} = l$ where $l \in \mathbb{R}$, or $l = \pm\infty$

then $\lim_{x \to \cdot} \frac{f(x)}{g(x)} = l$.

Taylor Polynomials

Idea: to use polynomials to approximate non-polynomial functions.

Suppose f is defined on an open interval I containing $a \in \mathbb{R}$, and f is differentiable ~~funct~~ as many times as we like on I i.e. for all $n \geq 1$, for all $x \in I$, $f^{(n)}(x)$ exists and is differentiable.

We want to approximate f near a using polynomials.

Case 1 $a=0$.

Consider the polynomial

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

with $c_i \in \mathbb{R}$.

Then

$$p'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots + nc_n x^{n-1}$$

$$p''(x) = 2c_2 + 3.2.c_3 x + 4.3.c_4 x^2 + \cdots + n(n-1)c_n x^{n-2}$$

$$p^{(3)}(x) = 3.2.1.c_3 + 4.3.2.c_4 x + \cdots + n(n-1)(n-2)c_n x^{n-3}$$

$\vdots$

Put $x=0$ then $p(0) = c_0 = 0!c_0$

$n! = n(n-1)(n-2)\ldots 2.1$

$0!$ defined to be 1

$$p'(0) = c_1 = 1!c_1$$

$$p''(0) = 2c_2 = 2!c_2$$

$$p^{(3)}(0) = 3.2.1.c_3 = 3!c_3$$

In general $p^{(k)}(0) = k!c_k$ (prove by induction)

So we can write

$$p(x) = p(0) + p'(0)x + \frac{p''(0)}{2!}x^2 + \frac{p^{(3)}(0)}{3!}x^3 + \cdots + \frac{p^{(n)}(0)}{n!}x^n .$$

We will consider a polynomial p to be a good approximation to f near 0 if

$$f(0) = p(0) \text{ and } f'(0) = p'(0) \text{ and } f^{(2)}(0) = p^{(2)}(0) \cdots \text{ and } f^{(n)}(0) = p^{(n)}(0).$$

Definition The Taylor polynomial $T_n(x)$ of order n for f about the point 0 is

$$T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n .$$

Note For all $0 \leq k \leq n$

$$T_n^{(k)}(0) = f^{(k)}(0).$$

Examples

1. Find the Taylor polynomials of orders 1, 2, ~~and~~ 3 and n about 0 for

$f(x) = e^x$.

Compute $f(0), f'(0), f''(0), f^{(3)}(0)$ and $f^{(n)}(0)$.

Since $f'(x) = e^x$

~~so~~
$$f(0) = e^0 = 1$$
$$f'(0) = e^0 = 1$$
$$f''(0) = e^0 = 1$$
$$f^{(3)}(0) = e^0 = 1$$
$$f^{(n)}(0) = e^0 = 1.$$

So

$$T_1(x) = f(0) + f'(0)x = 1 + x$$
$$T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$
$$= 1 + x + \frac{x^2}{2!}$$

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

2. $f(x) = \sin x$

$f'(x) = \cos x \quad f''(x) = -\sin x \quad f^{(3)}(x) = -\cos x$

$f^{(4)}(x) = \sin x$

So

$$f(0) = 0$$
$$f'(0) = 1$$
$$f''(0) = 0$$
$$f^{(3)}(0) = -1$$
$$f^{(4)}(0) = 0$$
$$\vdots$$

So

$$T_1(x) = 0 + 1.x = x$$

$$T_2(x) = 0 + 1.x + \frac{0}{2!}x^2 = x$$

$$T_3(x) = 0 + 1.x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3$$

$$= x - \frac{x^3}{3!}$$

In general if n is odd

$$\cancel{T_{2n}} \; T_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^{\frac{n-1}{2}}}{n!} x^n$$

and

$$T_{n+1}(x) = T_n(x).$$

<u>Case 2</u> $a \in \mathbb{R}$

i.e. use poly. to approximate f near a.

<u>Definition</u>

The <u>Tayler polynomial</u> of order n for f about the point a is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

<u>Check</u> For all $0 \le k \le n$

$$T_n^{(k)}(a) = f^{(k)}(a).$$

should perhaps be $T_{n,f,a}$ not T_n.

Example

The Taylor polynomial of order 3 for $f(x) = \sin x$ about $a = \frac{\pi}{3}$ is

$$T_3(x) = \sin\frac{\pi}{3} + \left(\cos\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + \frac{-\sin\frac{\pi}{3}}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{-\cos\frac{\pi}{3}}{3!}\left(x - \frac{\pi}{3}\right)^3$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{3}\right)^2 - \frac{1}{12}\left(x - \frac{\pi}{3}\right)^3.$$

Estimation of errors

Definition Let $T_n(x)$ be the Taylor polynomial of order n for f about a. The remainder term $R_n(x)$ is defined by

$$f(x) = T_n(x) + R_n(x)$$

ie. $$R_n(x) = f(x) - T_n(x).$$

Note $R_n(a) = f(a) - T_n(a) = 0$ so error term is 0 at $x = a$.

<u>Theorem</u> (Lagrange's form of the remainder)

For all $x \neq a$, there is a c between x and a so that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

<u>Note</u>

1. If $x < a$ then c is in (x, a)
   If $x > a$ then c is in (a, x).
2. The value of c depends on n (as well as on f, x and a).

<u>Proof</u> in case $a = 0$.

Fix $x \neq 0$. Define $g(t)$ by

$$g(t) = f(x) - \Big[f(t) + f'(t)(x-t) + \frac{f''(t)}{2!}(x-t)^2 + \cdots + \frac{f^{(n)}(t)}{n!}(x-t)^n + \frac{R_n(x)\,(x-t)^{n+1}}{x^{n+1}} \Big]$$

We will apply Rolle's Theorem to g, on the interval $[0, x]$ or $[x, 0]$.

$$g(0) = f(x) - [\underbrace{T_n(x)}_{\text{Taylor poly order } n \text{ for } f \text{ about } 0} + R_n(x)]$$ 9

put $t=0$

$$= f(x) - [T_n(x) + f(x) - T_n(x)]$$

$$= 0.$$

$$g(x) = f(x) - [f(x)] = 0.$$

put $t=x$

Thus by Rolle's Theorem, there is a c between 0 and x so that

$$g'(c) = 0.$$

Now differentiating w.r.t. t

$$g'(t) = -\Big[\cancel{f'(t)} + \cancel{f''(t)(x-t)} - \cancel{f'(t)}$$
$$+ \cancel{\frac{f^{(3)}(t)}{2!}(x-t)^2} - \cancel{f''(t)(x-t)}$$

erase

$$+ \cdots$$
$$+ \frac{f^{(n+1)}(t)}{n!}(x-t)^n - \cancel{\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}}$$
$$- \frac{(n+1)R_n(x)(x-t)^n}{x^{n+1}}\Big]$$

$$= \frac{f^{(n+1)}(t)}{n!}(x-t)^n - R_n(x)\frac{(n+1)(x-t)^n}{x^{n+1}}$$

So as $g'(c) = 0$

$$\frac{f^{(n+1)}(c)}{n!}(x-c)^n = R_n(x)\,\frac{(n+1)(x-c)^n}{x^{n+1}}$$

so

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\,x^{n+1}$$

is the remainder term at $a=0$.