Week 8 Lecture 2

Last time: defined the Taylor polynomial of order n for f about a is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The remainder term $R_n(x)$ is defined by

$$R_n(x) = f(x) - T_n(x)$$

and by a theorem, there is a c between x and a so that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

We have established

Taylor's formula For any $x \neq a$, there is a c between x and a so that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

~~When $n = 0$~~

When $n=0$

$$f(x) = f(a) + f'(c)(x-a)$$

i.e. $f'(c) = \dfrac{f(x)-f(a)}{x-a}$ for some c between x and a

This is the MVT.

<u>Example</u> Let $f(x) = \cos x$

a) Use the Taylor poly. for f of order 3 about 0 to estimate $\cos 0.5$.

b) Use the remainder term to specify an interval containing $\cos 0.5$.

a) $T_3(x) = 1 + 0x - \dfrac{x^2}{2!} + 0x^3$

$\quad\quad = 1 - \dfrac{x^2}{2}.$

so $\cos 0.5$ is approximately $T_3(0.5) = 0.875.$

b) $R_3(x) = \dfrac{f^{(4)}(c)}{4!}(x-0)^4$ for some c between 0 and x

so $R_3(0.5) = \dfrac{\cos c}{4!}(0.5)^4$, for some c between 0 and 0.5.

Now $0 < c < 0.5 < \frac{\pi}{2}$

so $0 < \cos c < 1$

so $0 < R_3(0.5) < \frac{1}{4!}(0.5)^4 \leq 0.003.$

Thus

$$f(0.5) = \cos 0.5 = T_3(0.5) + R_3(0.5)$$

Thus

$$0.875 < \cos 0.5 < 0.878.$$

<u>Example</u> What is the smallest n so that the Taylor polynomial of order n about 0 approximates the value of e to within 0.0005?

Let $f(x) = e^x$ then $f(1) = e^1 = e$.

Now

$$f(x) = e^x = T_n(x) + R_n(x)$$

so

$$e = T_n(1) + R_n(1).$$

Want to find smallest n so that

$$|R_n(1)| < 0.0005.$$

Now

$$\cancel{R_n(1) = \frac{f^{(n+1)}(c)}{(n+1)!}}$$

$$R_n(1) = \frac{f^{(n+1)}(c)}{(n+1)!}\, 1^{n+1}$$

$$= \frac{e^c}{(n+1)!}$$ for some c between 0 and 1.

So we want smallest n so that for all $0<c<1$

$$\frac{e^c}{(n+1)!} < 0.0005 .$$

~~Check: n=7~~. Since $0 < e^c < e$ we want smallest n s.t.

$$\frac{e}{(n+1)!} < 0.0005$$

Check: $n = 7$.

<u>Theorem</u> The Taylor polynomial $T_n(x)$ for f of order n about a is the unique polynomial of degree $\leq n$ so that

$$\lim_{x \to a} \frac{f(x) - T_n(x)}{(x-a)^n} = 0 .$$

~~Pr~~ Proof We first show

$$\lim_{x\to a} \frac{f(x)-T_n(x)}{(x-a)^n} = 0.$$

Now $f(x)-T_n(x) = R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

for some c between x and a.

The function $f^{(n+1)}$ is continuous since by assumption it is differentiable. Hence on any closed and bounded interval containing a, ~~then~~ the function $f^{(n+1)}$ is bounded.

So there is some K (depending on x) so that for all c between a and x

$$\left|\frac{f^{(n+1)}(c)}{(n+1)!}\right| \le K.$$

Thus for all $x \neq a$,

$$0 \le \left|\frac{f(x)-T_n(x)}{(x-a)^n}\right| \le K|x-a|.$$

So

$$-K|x-a| \le \frac{f(x)-T_n(x)}{(x-a)^n} \le K|x-a|.$$

So by Squeeze Law

$$\lim_{x \to a} \frac{f(x) - T_n(x)}{(x-a)^n} = 0.$$

Uniqueness?

Suppose p is a polynomial of degree $\leq n$ so that

$$\lim_{x \to a} \frac{f(x) - p(x)}{(x-a)^n} = 0.$$

We will show that $p = T_n$.

Now

$$\lim_{x \to a} \frac{T_n(x) - p(x)}{(x-a)^n} = \lim_{x \to a} \left(\frac{f(x) - p(x)}{(x-a)^n} - \frac{f(x) - T_n(x)}{(x-a)^n} \right)$$

$= 0.$ since both limits exist and equal 0.

Let $q(x) = T_n(x) - p(x)$.

Then q is a polynomial of degree $\leq n$.

Thus

$$q = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n \qquad c_i \in \mathbb{R}$$

Replace x by $a + (x-a)$, to get

$$q(x) = d_0 + d_1 (x-a) + d_2 (x-a)^2 + \cdots + d_n (x-a)^n \qquad d_i \in \mathbb{R}.$$

If q is not the zero polynomial then there is a smallest k so that $d_k \neq 0$. Then

$$q(x) = d_k(x-a)^k + \cdots + d_n(x-a)^n$$
$$= (x-a)^k \left[d_k + \cdots + d_n(x-a)^{n-k} \right]$$
with $d_k \neq 0$.

Hence

$$\lim_{x \to a} \frac{T_n(x) - p(x)}{(x-a)^k} = \lim_{x \to a} \frac{q(x)}{(x-a)^k}$$
$$= \lim_{x \to a} \frac{(x-a)^k \left[d_k + \cdots + d_n(x-a)^{n-k} \right]}{(x-a)^k}$$
$$\cancel{= \lim_{x \to a} \frac{d_k + \cdots + d_n(x-a)^{n-k}}{(x-a)^{n-k}}}$$
$$= \lim_{x \to a} \left[d_k + d_{k+1}(x-a) + \cdots + d_n(x-a)^{n-k} \right]$$
$$= d_k \neq 0.$$

But

$$\lim_{x \to a} \frac{T_n(x) - p(x)}{(x-a)^k} = \lim_{x \to a} \frac{T_n(x) - p(x)}{(x-a)^n} (x-a)^{n-k}$$
$$= \left(\lim_{x \to a} \frac{T_n(x) - p(x)}{(x-a)^n} \right) \left(\lim_{x \to a} (x-a)^{n-k} \right)$$
$$= 0 \cdot 0 = 0$$

so we get a contradiction since $d_k \neq 0$. Therefore q is the zero poly. (8)

so $T_n(x) = p(x)$.

New Taylor polynomials from old

Example Find the Taylor poly of order $2n$ for ~~f(x) = e~~ $f(x) = e^{x^2}$

about 0.

We know the Taylor poly. of order n for

$g(x) = e^x$ about 0 is

$$T_n'(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}.$$

Claim: Taylor poly. for e^{x^2} is

$$T_{2n}(x) = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots + \frac{x^{2n}}{n!}$$

Proof: by Theorem, enough to show

$$\lim_{x \to a} \frac{f(x) - T_{2n}(x)}{(x-a)^{2n}} = 0 \quad \text{with } a = 0.$$

$$\lim_{x\to 0} \frac{e^{x^2} - T_{2n}(x)}{x^{2n}}$$

Next time. (sub in $x=0$, use l'Hôpital probably works)

Example

Estimating integrals.

~~Using~~ Use the Taylor polynomial of order 5 for $\sin x$ about 0 to find the Taylor polynomial of order 10 for $\sin x^2$ about 0 and hence estimate

$$\int_0^1 \sin(x^2)\,dx.$$

We have the Taylor poly. order 5 for $\sin x$ about 0 is $T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$.

So it can be proved using the prev. Thm that the Taylor poly. of order 10 for $\sin(x^2)$ is

$$T_{10}(x) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!}.$$

So there is a c between 0 and x

so that

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \underbrace{\frac{(-\sin c)}{6!}x^{12}}_{\text{from Lagrange's form of remainder}}.$$

Thus

$$\int_0^1 \sin x^2\,dx = \overbrace{\int_0^1 x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!}\,dx}^{\cong 0.3103}$$

$$+ \int_0^1 \frac{-\sin c}{6!}x^{12}\,dx$$

Now

$$\left|\int_0^1 \frac{-\sin c}{6!}x^{12}\,dx\right| \le \int_0^1 \left|\frac{-\sin c}{6!}x^{12}\right|dx$$

$$\le \frac{1}{6!}\int_0^1 x^{12}\,dx$$

$$\le 0.0001$$

So $\int_0^1 \sin x^2\,dx$ lies in interval

$[0.31017, 0.31019]$.