Week 9 Lecture 1

From last time:

to show that Taylor polynomial of order $2n$ about 0 for $g(x) = e^{x^2}$ is

$$p(x) = 1 + x^2 + \frac{x^4}{2!} + \cdots + \frac{x^{2n}}{n!}$$

i.e. sub in x^2 for x in Taylor poly. of order n about 0 for $f(x) = e^x$

it suffices to show

$$\lim_{x \to 0} \frac{e^{x^2} - p(x)}{x^{2n}} = 0.$$

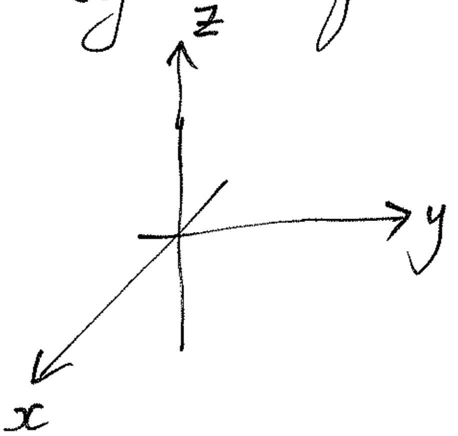
To prove this limit, use l'Hôpital's rule $2n$ times.

Curves and Surfaces in 3-dimensional space

Notation $\mathbb{R}^2 = \{ \underbrace{(x,y)}_{\text{ordered pair}} \mid x, y \in \mathbb{R} \}$

$\mathbb{R}^3 = \{ \overbrace{(x,y,z)}^{\text{ordered triple}} \mid x, y, z \in \mathbb{R} \}$

To draw things in $\mathbb{R}^3$ we use axes given by "right-hand screw rule"

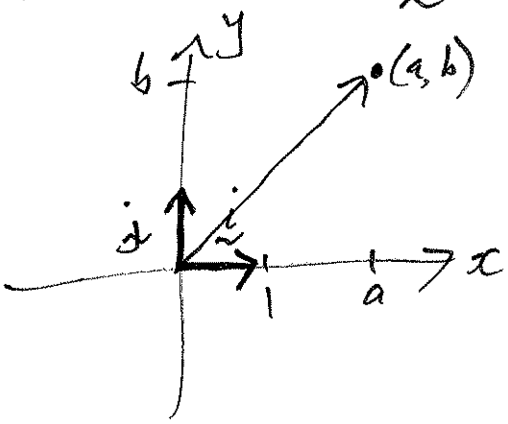


put fingers of your right hand along positive x-axis (the one with the arrow)

then thumb along positive z-axis, then if curl in fingers, get positive y-axis.

Vector notation

in $\mathbb{R}^2$ $\quad \underset{\sim}{i} = (1,0) \quad \underset{\sim}{j} = (0,1)$

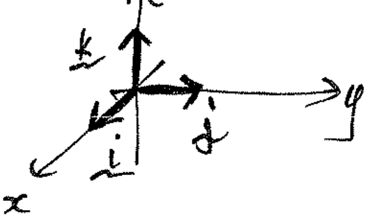


then

$$\begin{aligned}(a,b) &= a\underset{\sim}{i} + b\underset{\sim}{j} \\ &= a(1,0) + b(0,1) \\ &= (a,0) + (0,b) \\ &= (a,b).\end{aligned}$$

in $\mathbb{R}^3$ $\quad \underset{\sim}{i} = (1,0,0) \quad \underset{\sim}{j} = (0,1,0) \quad \underset{\sim}{k} = (0,0,1)$

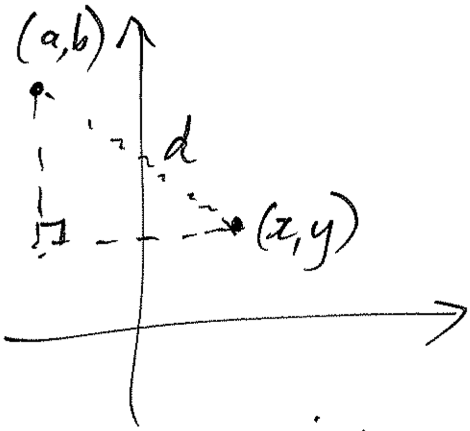
$(a,b,c) = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$



Distances in $\mathbb{R}^2$ and $\mathbb{R}^3$

If (x,y) and (a,b) are points in $\mathbb{R}^2$, the distance between them is

$$d=\sqrt{(x-a)^2+(y-b)^2}$$



(Pythagoras)

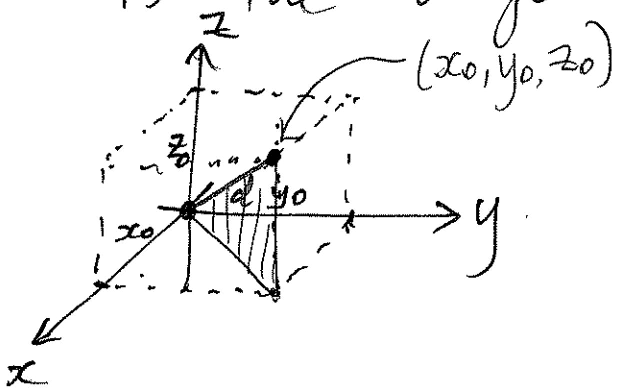
We'll write

$|(x,y)-(a,b)|$ for this distance.

In $\mathbb{R}^3$: the distance between (x,y,z) and (a,b,c) is

$$d=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$$

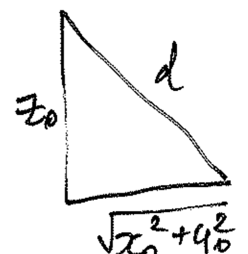
Why? If $x_0, y_0, z_0 > 0$ and $(a,b,c)=(0,0,0)$ is the origin



d is length of the long diagonal of this box

So by Pythagoras' rule

$$d = \text{length of hypotenuse of shaded triangle}$$



$$= \sqrt{\left(\sqrt{x_0^2+y_0^2}\right)^2 + z_0^2}$$

$$= \sqrt{x_0^2 + y_0^2 + z_0^2}\,.$$

$$= \sqrt{(x_0-0)^2 + (y_0-0)^2 + (z_0-0)^2}\,.$$

<u>Parametric description of curves in $\mathbb{R}^2$ and $\mathbb{R}^3$</u>

Roughly speaking, a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ is a subset which locally looks like a line.

Let I be an interval in $\mathbb{R}$.

A function $\phi : I \longrightarrow \mathbb{R}^2$ describes a curve in $\mathbb{R}^2$, and a function $\psi : I \longrightarrow \mathbb{R}^3$ describes a curve in $\mathbb{R}^3$.

Examples

1. Let $I = [0, 2\pi]$ and let

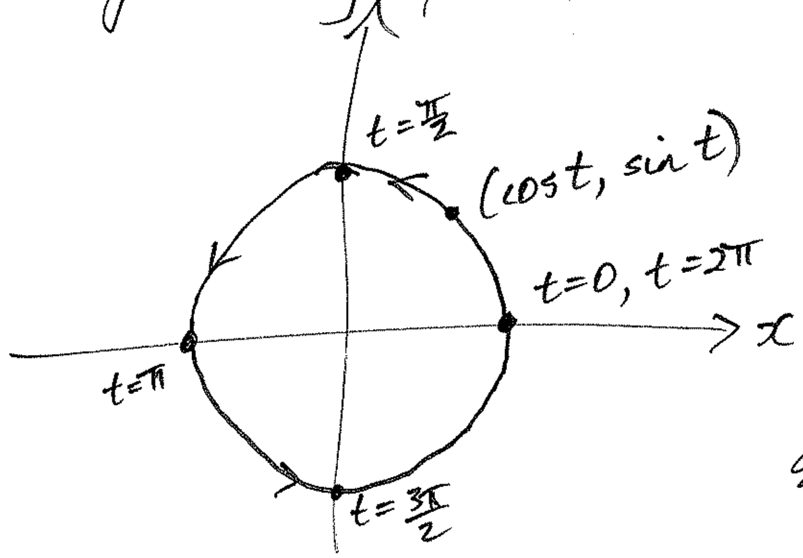
$$\phi : I \to \mathbb{R}^2$$

be given by

$$\phi(t) = (\cos t, \sin t)$$

(think of t as time)

As the parameter t varies, what is the image of ϕ?



$t=0 \qquad \phi(0) = (1,0)$

$t=\frac{\pi}{2} \qquad \phi\left(\frac{\pi}{2}\right) = (0,1)$

$t=\pi \qquad \phi(\pi) = (-1,0)$

$t=\frac{3\pi}{2} \qquad \phi\left(\frac{3\pi}{2}\right) = (0,-1)$

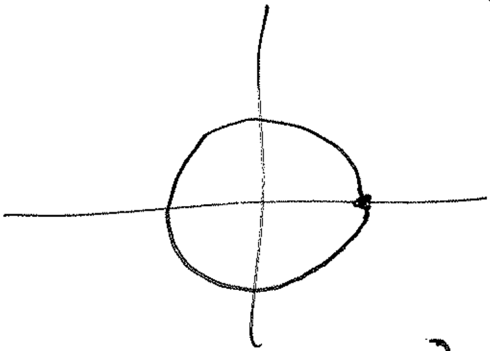
$t=2\pi \qquad \phi(2\pi) = (1,0)$

The curve traces out the unit circle in $\mathbb{R}^2$ once.

2. If $I = \mathbb{R}$ and

$$\phi(t) = (\cos t, \sin t)$$

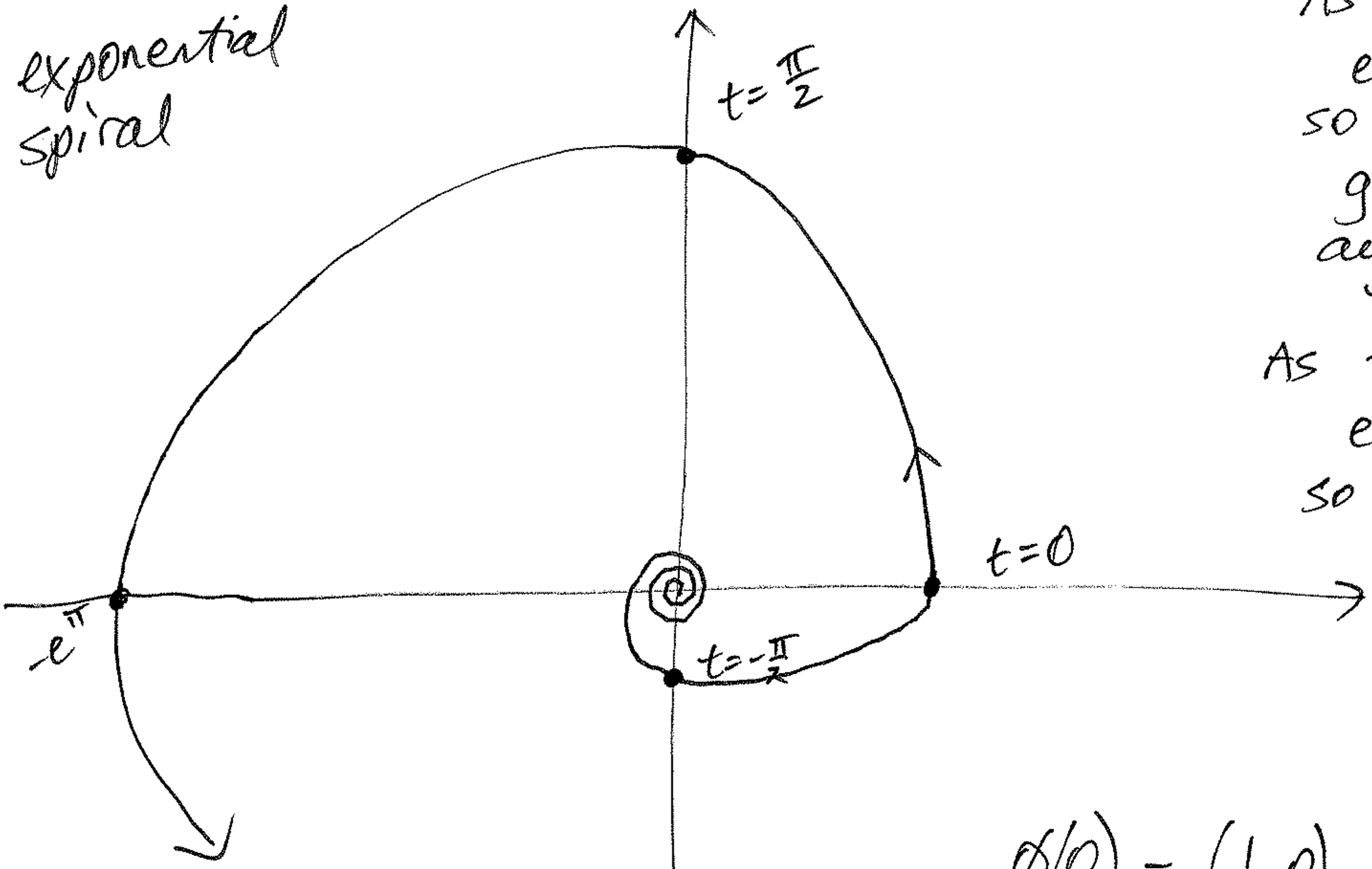
then the curve wraps around the unit circle infinitely many times



3. Let $\phi : \mathbb{R} \to \mathbb{R}^2$ be given by

$$\phi(t) = (e^t \cos t, e^t \sin t).$$

exponential spiral



As $t \to \infty$
$e^t \to \infty$
so $\phi(t)$ goes far away from the origin

As $t \to -\infty$
$e^t \to 0^+$
so $\phi(t)$ approaches the origin.

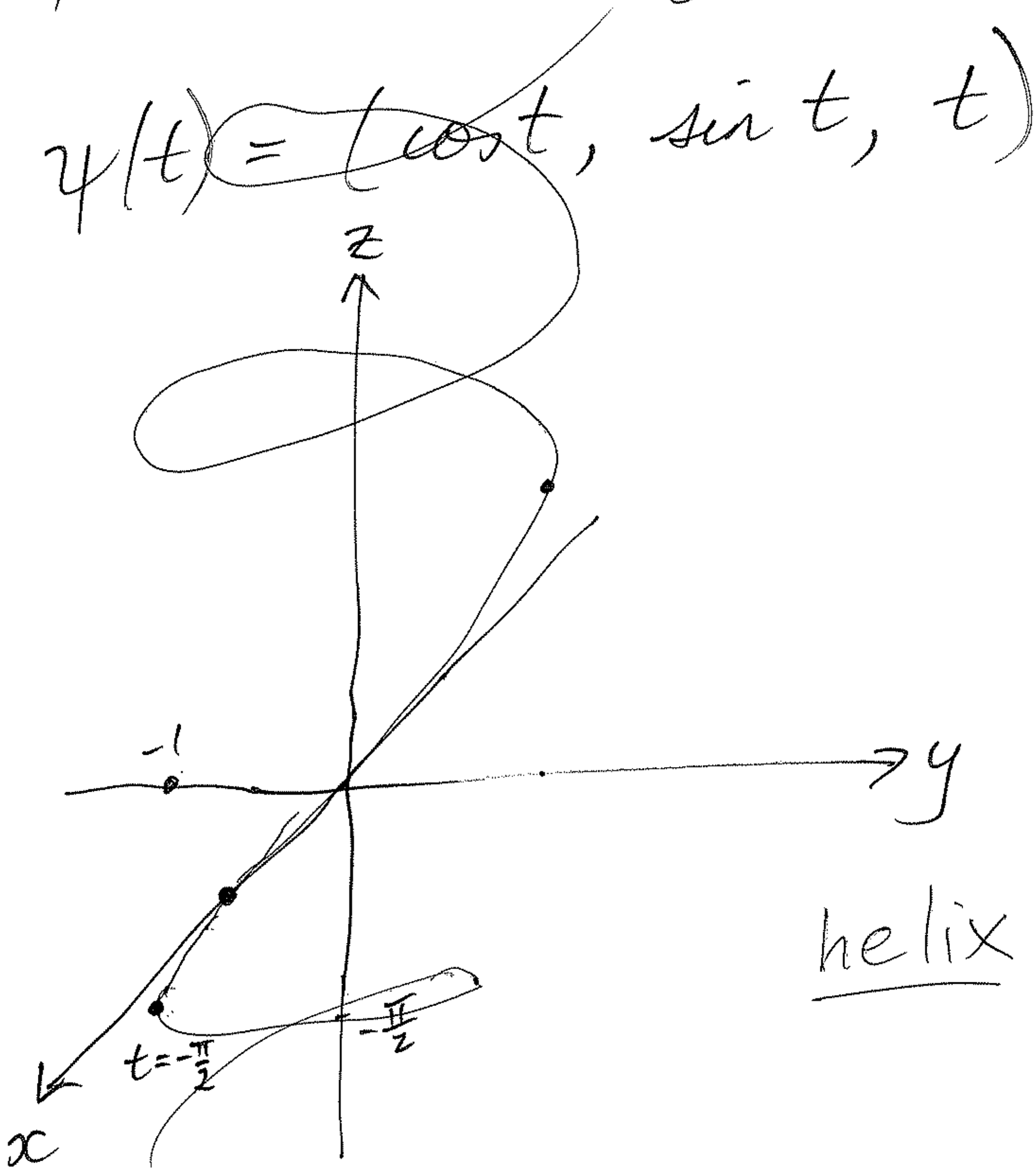
$\phi(0) = (1, 0)$

$\phi(\frac{\pi}{2}) = (0, e^{\frac{\pi}{2}})$

$\phi(-\frac{\pi}{2}) = (0, e^{-\frac{\pi}{2}})$

4. Let $\psi : \mathbb{R} \to \mathbb{R}^3$ be given by (7)

$$\psi(t) = (\cos t, \sin t, t)$$



helix

$$\phi\left(-\frac{\pi}{2}\right) = \left(0, -1, -\frac{\pi}{2}\right)$$

$$\phi(0) = (1,0,0)$$

$$\phi\left(\frac{\pi}{2}\right) = \left(0, 1, \frac{\pi}{2}\right)$$

Parametric curves eg. $\phi(t) = (\cos t, \sin t)$

or $\psi(t) = (\cos t, \sin t, t)$ give

parametric equations

$\mathbb{R}^2$ $x = \cos t$ $y = \sin t$

$\mathbb{R}^3$ $x = \cos t$ $y = \sin t$ $z = t$.

Surfaces

Roughly speaking, a surface is a subset of $\mathbb{R}^3$ which locally looks like a plane.

Planes in $\mathbb{R}^3$

A *plane* in $\mathbb{R}^3$ is a set of points of the form

$$\{ (x, y, z) \mid ax + by + cz = d \}$$

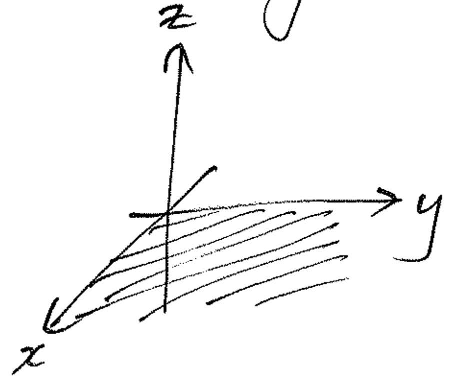
where $a, b, c, d \in \mathbb{R}$ are constants and at least one of a, b, c is non-zero.

This plane has equation $ax + by + cz = d$.

Examples

1. $\{(x,y,z) \mid z=0\}$ $\quad a=0, b=0, c=1, d=0$

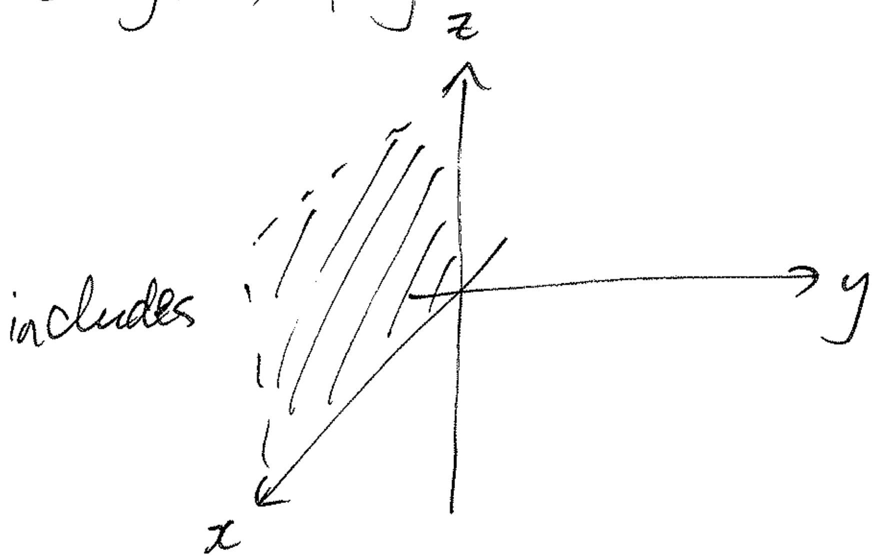
This is the xy-plane i.e. horizontal plane through the origin



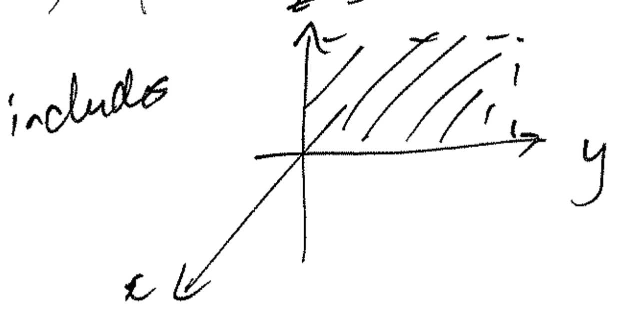
In first octant i.e. $x,y,z \geq 0$

i.e. the xy-plane in $\mathbb{R}^3$ has equation $z=0$.

2. $\{(x,y,z) \mid y=0\}$ is the xz-plane

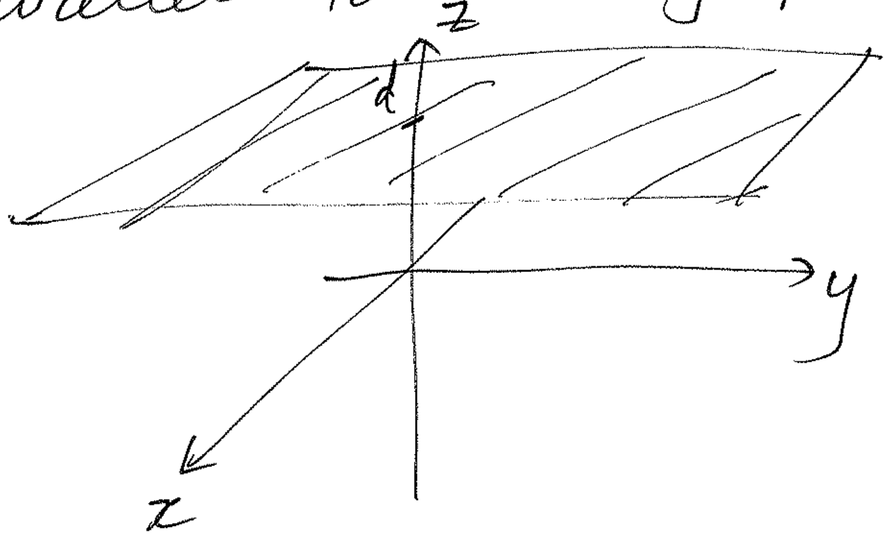


3. $\{(x,y,z) \mid x=0\}$ is the yz-plane



4. $\{ (x,y,z) \mid z = d \}$

is the horizontal plane at height d (parallel to the xy-plane)



Spheres

A sphere in $\mathbb{R}^3$ with centre (a,b,c) and radius r is given by the set of points

$$\{ (x,y,z) \mid \sqrt{(x-a)^2+(y-b)^2+(z-c)^2} = r \}$$

ie. all points at distance r from (a,b,c).

Example The unit sphere centred at the origin is $\{ (x,y,z) \mid x^2+y^2+z^2=1 \}$.

Cylinders

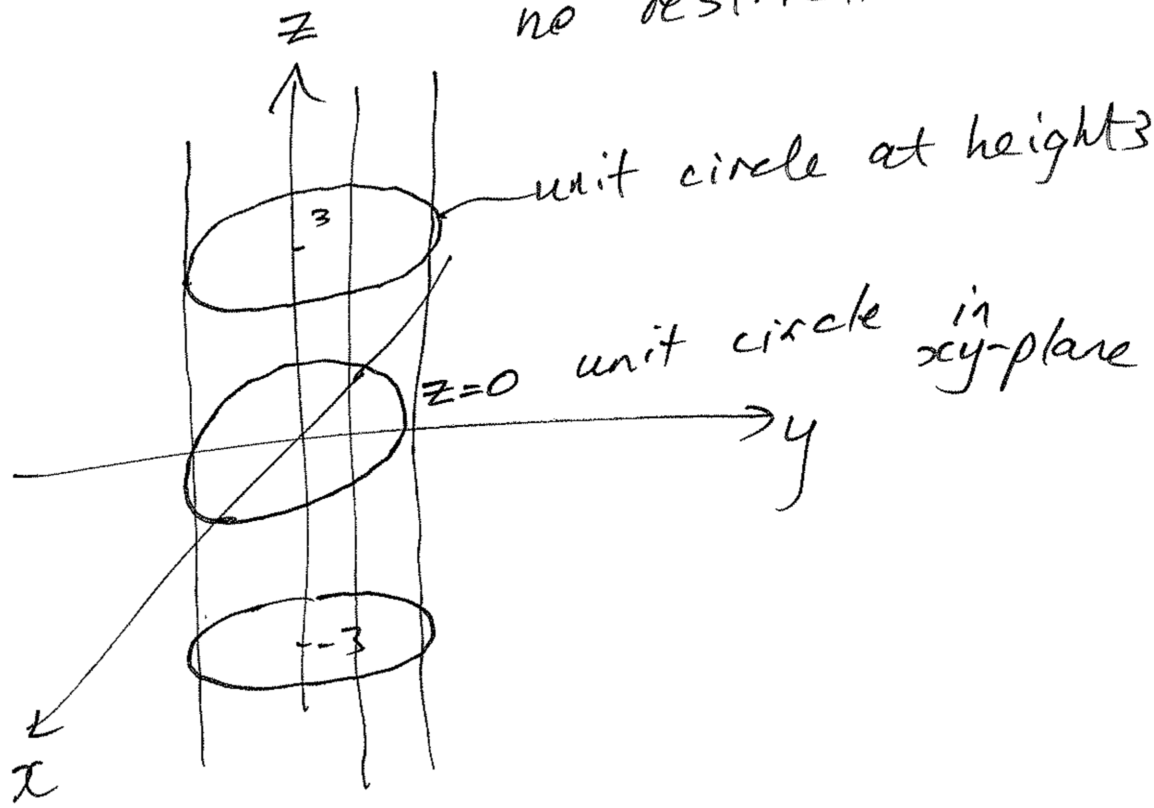
A cylinder is a surface which is a union of parallel lines, all perpendicular to a given curve.

Examples

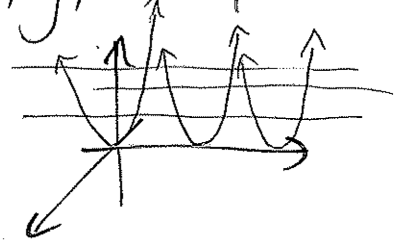
1. Circular cylinder

$$\{(x,y,z) \mid x^2+y^2=1\}$$

no restriction on z



2. $\{(x,y,z) \mid z=x^2\}$ parabolic cylinder



$z=x^2$ is a parabola in xz-plane