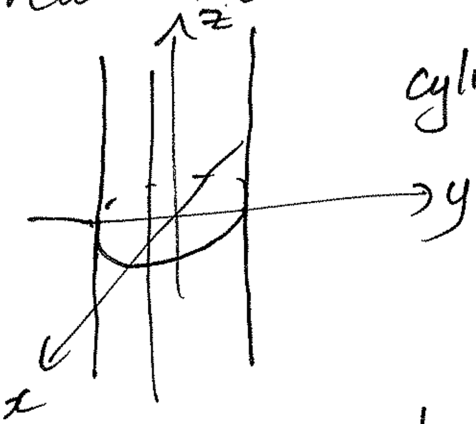
Week 9 Lecture 2

For this part of course, Stewart Calculus is a good resource.

When we have a set of points in $\mathbb{R}^3$

e.g. $\{(x,y,z) \mid x^2+y^2=1\}$

~~this mea~~ the fact that z doesn't appear after the $\mid$ means z takes all real values.



cylinder

e.g. $\{(x,y,z) \mid z=0\}$

here x and y are unrestricted so they take all real values so we get everything in the xy-plane

Definition A real-valued function of two variables is a function

$$f : D \to \mathbb{R}$$

where the domain D is a subset of $\mathbb{R}^2$ (possibly all of $\mathbb{R}^2$).

Examples

1. The volume V of a cylinder of height h and radius r is

$$V(r,h) = \pi r^2 h .$$

Domain: $h > 0$, $r > 0$.

2. Cobb-Douglas production function

$$P(K,L) = b L^{\alpha} K^{1-\alpha}$$

where P is production
K capital
L labour

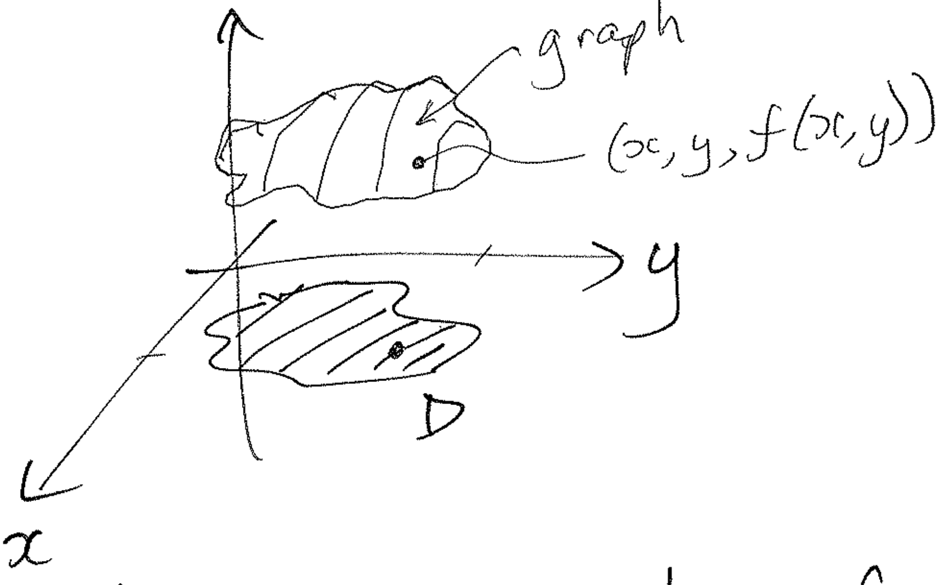
and b, α constants $b \geq 1$, $0 < \alpha < 1$.

Graphs

The *graph* of $f : D \to \mathbb{R}$ is the subset of $\mathbb{R}^3$ given by

$$\{ (x, y, z) \mid z = f(x,y), \ (x,y) \in D \}$$

This is a surface in $\mathbb{R}^3$ situated vertically above or below the domain D.



Not all surfaces are graphs of a single function.

e.g. unit sphere $\{ (x,y,z) \mid x^2+y^2+z^2=1 \}$

The equation $\underline{z^2 = 1 - x^2 - y^2}$ has 2 solutions $z = \pm\sqrt{1-x^2-y^2}$ so there is no single $f : D \to \mathbb{R}$ so that the ~~set~~ graph

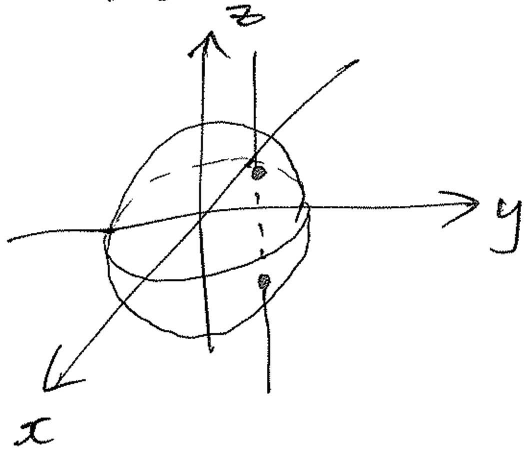
$$\{ (x,y,z) \mid z = f(x,y) \}$$

gives the whole sphere.

Vertical ~~test~~ line test

A surface in $\mathbb{R}^3$ is the graph of $z = f(x,y)$ if and only if every line parallel to the z-axis (i.e. every vertical line) intersects the surface at most once.

e.g. sphere



not the graph of a function.

Domain and range

Examples

1. Let $f(x,y) = \sqrt{1-x^2-y^2}$.
Find the natural domain of f and describl the surface $z = f(x,y)$.

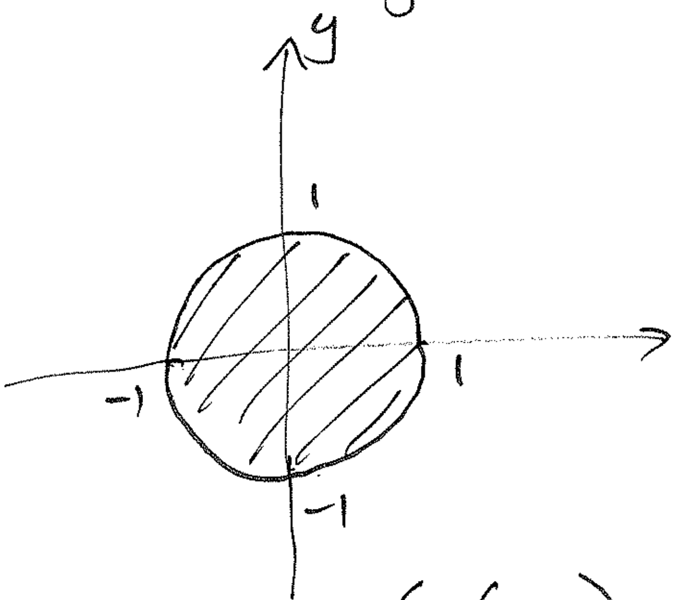
Domain: all (x,y) so that f is defined at (x,y).

$$f(x,y) = \sqrt{1-x^2-y^2}$$

so we must have

$$1-x^2-y^2 \geq 0$$

so $\quad x^2+y^2 \leq 1$.



Domain is $\{(x,y) \mid x^2+y^2 \leq 1\}$.

Range? $\quad 0 \leq 1-x^2-y^2 \leq 1$

since $x^2 \geq 0,\ y^2 \geq 0$

so range is

$$0 \leq f(x,y) = \sqrt{1-x^2-y^2} \leq 1$$

i.e. $[0,1]$.

Surface? $\quad z = \sqrt{1-x^2-y^2}$

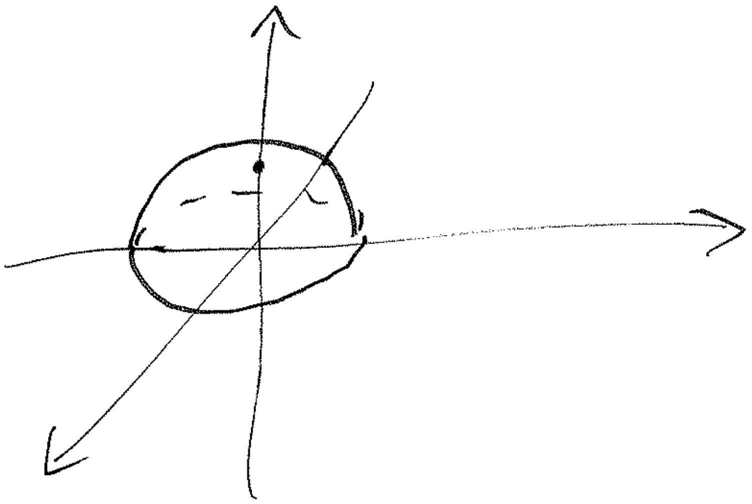
Square both sides

$$z^2 = 1 - x^2 - y^2$$

$$x^2 + y^2 + z^2 = 1$$ equation of unit sphere

The graph is the top half of the unit sphere

top $\rightarrow$ z is positive square root.



2. Find and sketch the domain D of

$$f(x,y) = \ln(\sqrt{x+y} - 1)$$

Note : D is a subset of $\mathbb{R}^2$, D is in the xy-plane.

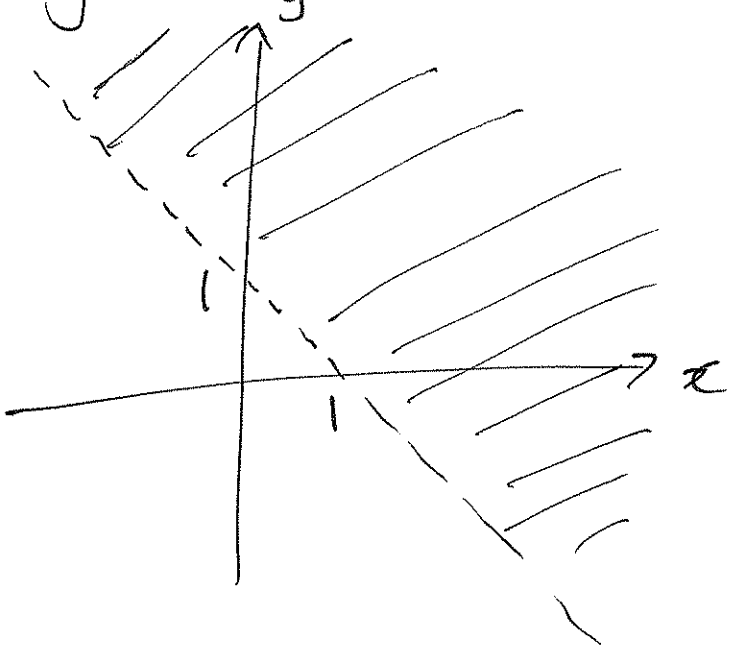
Domain of ln is $(0, \infty)$ so

$$\sqrt{x+y} - 1 > 0 \qquad (x+y \geq 0)$$

$$\sqrt{x+y} > 1$$

$$x + y > 1$$ (this is stronger than

i.e. $y > 1 - x$

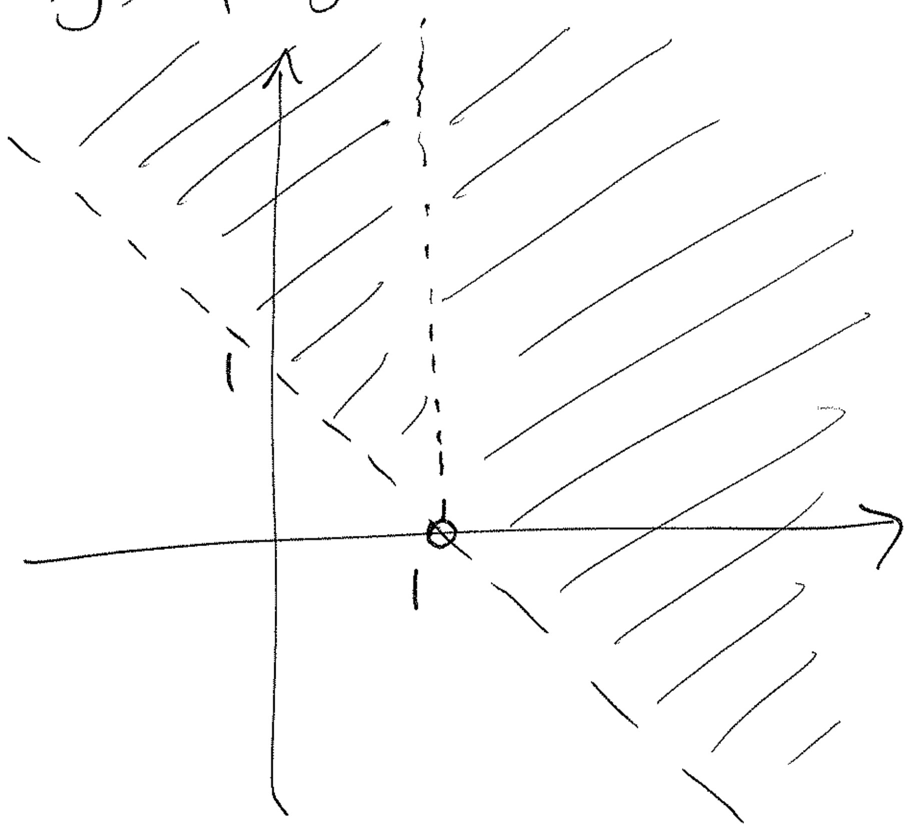


all points above line $y = 1 - x$.

3. Find and sketch domain, D, of

$$g(x,y) = \frac{\ln(\sqrt{x+y} - 1)}{x-1}$$

$$D = \{ (x,y) \mid y > 1 - x \ , \ x \neq 1 \}$$



4. Find domain and range of

$$h(x,y) = \frac{x^2 - y^2}{x^2 + y^2}.$$

Domain : $\{ (x,y) \mid x^2+y^2 \neq 0 \}$

$= \{ (x,y) \mid (x,y) \neq (0,0) \}$

i.e. all points except the origin.

Range : use polar coordinates

$x = r\cos\theta \qquad y = r\sin\theta$

Then $\quad h(x,y) = \dfrac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$

$= \cos^2\theta - \sin^2\theta$

(since $\cos^2\theta + \sin^2\theta = 1$)

$= \cos 2\theta.$

Now $\quad -1 \leq \cos 2\theta \leq 1 \quad$ as θ varies

so the range of h is $[-1, 1]$.

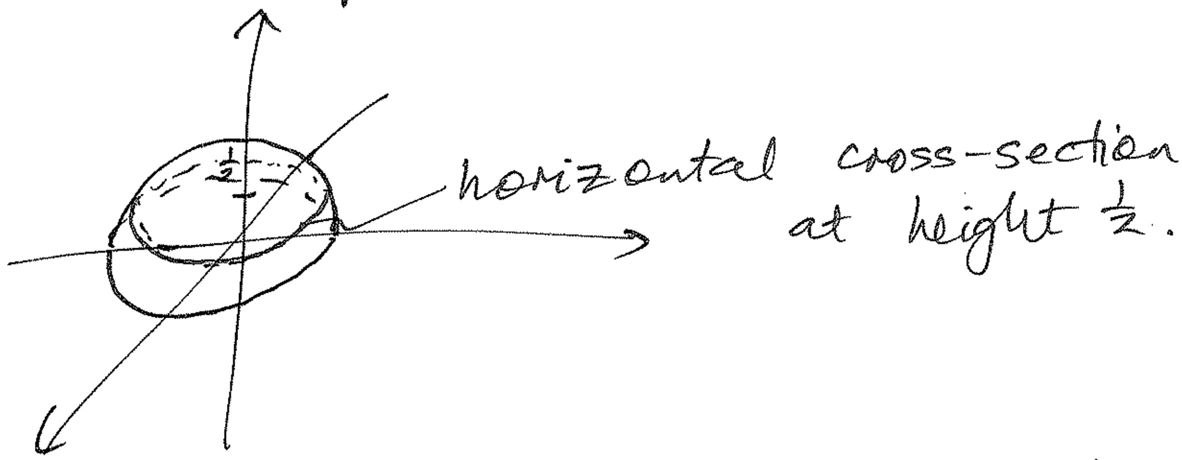
Level Curves

Suppose a surface is given by

$$z = f(x,y).$$

The <u>horizontal cross-section at height c</u> is the intersection of the surface with the horizontal plane $z = c$.

e.g. on unit hemi sphere



The <u>level curve of f at height c</u> is the curve in the xy-plane obtained by projecting the horizontal cross-section at height c to the xy-plane.

Examples

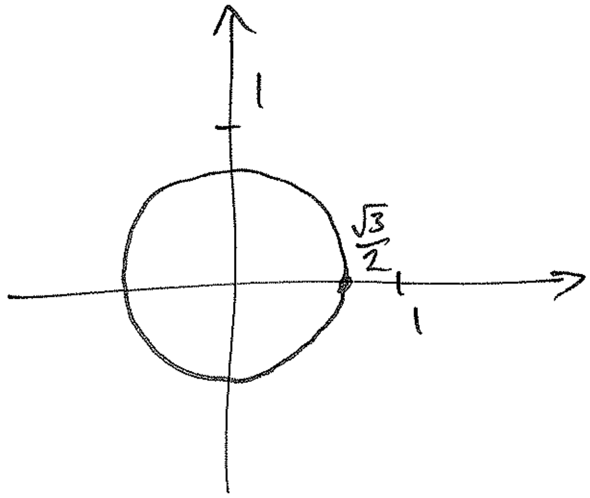
1. If $f(x,y) = \sqrt{1-x^2-y^2}$

then the level curve at height $\frac{1}{2}$

is $\{(x,y) \mid \sqrt{1-x^2-y^2} = \frac{1}{2}\}$

$= \{(x,y) \mid x^2+y^2 = \frac{3}{4}\}$

so we get the circle centre $(0,0)$
radius $\frac{\sqrt{3}}{2}$.



Plotting level curves for various values of c, usually evenly spaced, gives a <u>contour map</u> of the surface.

Contour lines close together means surface is steeper.

Example

$$z = x^2 + y^2$$

i.e. graph of $f(x,y) = x^2 + y^2$.

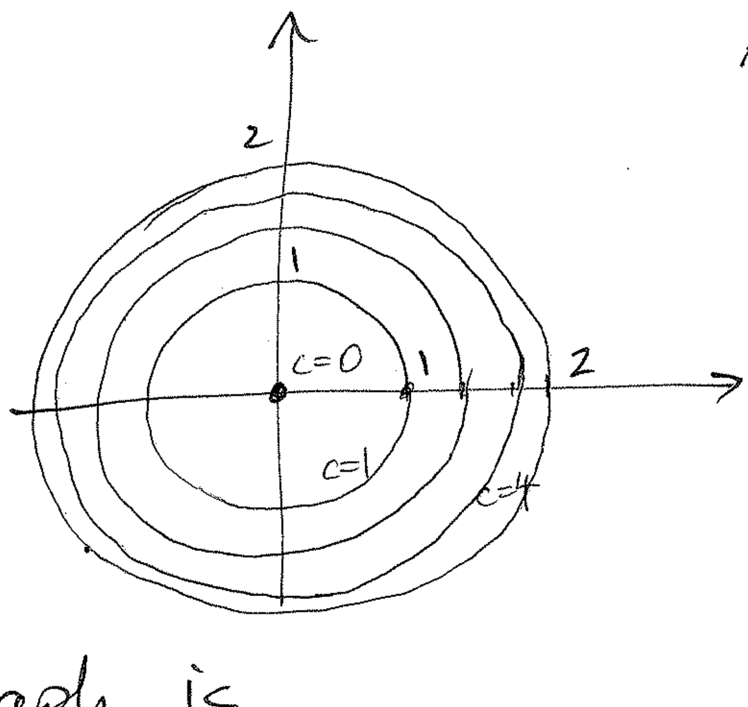
Domain of f is all of $\mathbb{R}^2$

Range of f is $[0, \infty)$.

Level curves? $x^2 + y^2 = c$

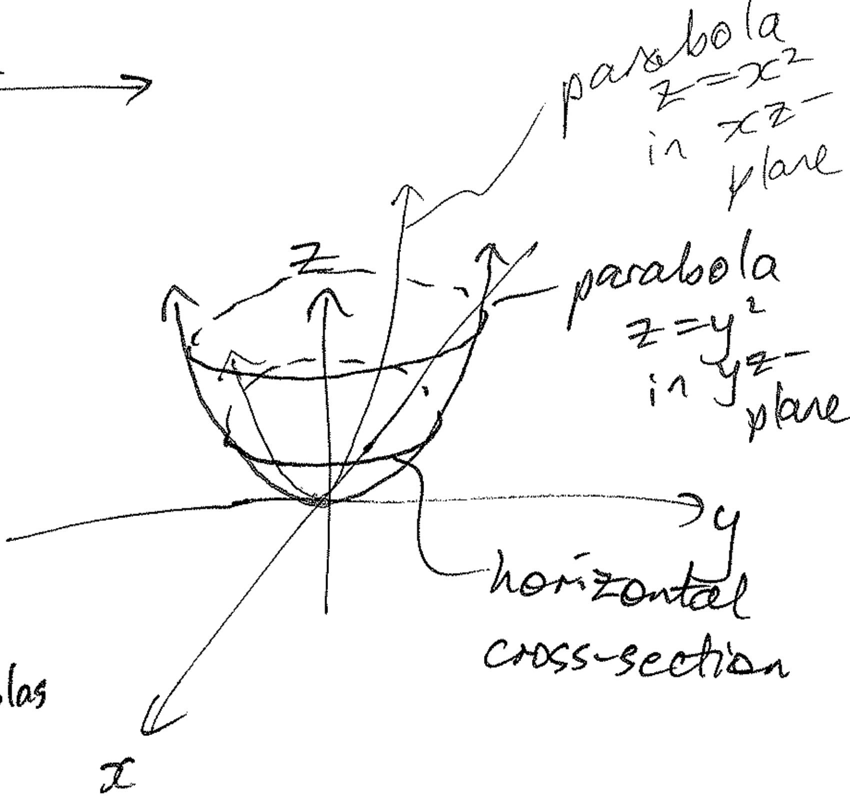
Circle centre $(0,0)$, radius $\sqrt{c}$

(with $c \geq 0$)



As c increases, circles get closer together

graph is paraboloid



If $y=0$ $\quad z = x^2$
If $x=0$ $\quad z = y^2$
$\Big\}$ parabolas

2. $h(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$

range is $[-1, 1]$, domain is all $(x,y) \neq (0,0)$

Plot the level curves at heights

$c=-1 \quad c=-\frac{1}{2} \quad c=0 \quad c=\frac{1}{2} \quad c=1.$

At height $c=-1$

$$\frac{x^2-y^2}{x^2+y^2} = -1$$

$$x^2-y^2 = -x^2-y^2$$

$$2x^2=0$$

$$x^2=0$$

$$x=0$$

At $c=-\frac{1}{2}$ $\quad y = \pm\sqrt{3}x$

$c=0$ $\quad x = \pm y$

$c=\frac{1}{2}$ $\quad y = \pm\frac{1}{\sqrt{3}}x$

$c=1$ $\quad y=0.$