The Banach-Tarski Paradox and Amenability

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Honours Course, Semester 2 2012

31 July 2012

1 Course description

A form of the Banach–Tarski paradox is the following bizarre statement:

> A sphere can be cut into finitely many pieces which can be put back together to create two spheres, each the same size as the original one.

We will begin by making precise this sort of statement, and then state and prove the Banach–Tarski paradox. One main ingredient of the proof is the Axiom of Choice, and the other is the fact that a free group on two generators does not satisfy a property called *amenability*. Roughly speaking, a group G is amenable if there is a measure on the set of bounded functions on G that is invariant under translation by group elements. We will prove the equivalence of this definition and several other formulations of amenability, involving the large-scale geometry of Cayley graphs, fixed point properties and unitary representations. We will then be able to establish the main examples of amenable and non-amenable groups, and to consider the relationship between the amenability of G and of its subgroups, quotients and extensions.

2 Assessment

Three assignments each worth 10%, due at the end of weeks 4, 9 and 13, and one closed-book exam worth 70%.

3 Prerequisites

This course will involve group theory, measure theory and functional analysis. Thus it is strongly recommended that students have taken:

- MATH2968 Algebra (Advanced)
- MATH3969 Measure Theory and Fourier Analysis (Advanced)
- the first semester of Honours Functional Analysis

4 Course outline

Review and overview (1 lecture)

The Banach-Tarski Paradox (5 lectures)

- Definition of paradoxical decompositions of sets and of paradoxical groups. Role of the Axiom of Choice.
- Group presentations and free groups. Proof that F_2 is paradoxical.
- Review of $O(n)$, $SO(n)$ and their actions on $\mathbb{R}^n$ and the sphere. Proof that $SO(n)$ has an $\mathbb{F}_2$ subgroup for $n \geq 3$.
- Statement and proof of versions of the Banach–Tarski paradox.
- Relationship between paradoxical decompositions and invariant measures.

Invariant Means (6 lectures)

- Discrete groups and locally compact groups: definitions and examples
- Review of measure theory and integration
- Haar measure: existence and uniqueness, examples
- Review of functional analysis
- Invariant means for discrete groups and locally compact groups
- Application: amenability of finite groups and compact groups. Examples.

Følner Conditions (8 lectures)

- Cayley graphs
- Quasi-isometries
- Følner conditions for finitely generated, discrete and locally compact groups
- Application: amenability of abelian groups, non-amenability of free groups
- Amenability and growth conditions
- Equivalence of amenability and the existence of a Følner sequence, via Reiter's Property

Amenability and group theory (3 lectures)

- Review of normal subgroups, quotient groups and group extensions
- Proof that the class of amenable groups is closed under taking subgroups, quotients and group extensions, using fixed point properties and unitary representations
- Application: solvable groups are amenable.
- The so-called von Neumann conjecture and counterexamples, the Tits Alternative and the Ruziewicz Problem.

Summary and review (1 lecture)

References

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[L] A. Lubotzky, *Discrete Groups, Expanding Graphs and Invariant Measures*, Birkhäuser, Basel, 1994.

[WM] D. Witte Morris, *Introduction to Arithmetic Groups*, `http://arxiv.org/abs/math/0106063v4`

[Pat] A.T. Paterson, *Amenability*, American Mathematical Society, Providence, 1988.

[Pi] J.-P. Pier, *Amenable locally compact groups*, Wiley, New York, 1984.

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[W] S. Wagon, *The Banach–Tarski paradox*, Cambridge University Press, Cambridge, 1985.

[Z] R.J. Zimmer, *Ergodic Theory and Semisimple Groups*, Birkhäuser, Boston, 1984.