The Banach–Tarski Paradox and Amenability
Lecture 11: Invariant Means

30 August 2012

Essentially bounded functions on groups

Let G be a locally compact group with Haar measure μ.

Suppose $f : G \to \mathbb{R}$ is measurable. (That is, for every open set $U \subset \mathbb{R}$, $f^{-1}(U)$ is a Borel set of G.)

A point $y \in \mathbb{R}$ is in the essential range of f if for all open neighbourhoods U of y, $\mu(f^{-1}(U)) > 0$. The function f is essentially bounded if the essential range is a bounded set.

Example

If G is a countable group and μ is counting measure, then every function $f : G \to \mathbb{R}$ is measurable, the essential range is the same as the range, and a function $f : G \to \mathbb{R}$ is essentially bounded if and only if it is bounded. In particular if G is finite then every function $f : G \to \mathbb{R}$ is bounded.

The space $\mathcal{L}^\infty(G)$

Let

$$\mathcal{L}^\infty(G) = \{f : G \to \mathbb{R} : f \text{ is measurable and essentially bounded}\}$$

Then $\mathcal{L}^\infty(G)$ is a vector space over $\mathbb{R}$. We define

$$\|f\|_\infty = \sup\{|x| : x \text{ is in the essential range of } f\}$$

Example

If G is a countable group and μ is counting measure, and $f : G \to \mathbb{R}$ is bounded, then

$$\|f\|_\infty = \sup\{|f(g)| : g \in G\}$$

In particular if G is finite then

$$\|f\|_\infty = \max\{|f(g)| : g \in G\}$$

The space $L^\infty(G)$

We form the quotient vector space

$$L^\infty(G) = \mathcal{L}^\infty(G)/\{f \in \mathcal{L}^\infty(G) \mid f = 0 \ \mu\text{–a.e.}\}$$

This is a Banach space. By abuse of notation the norm is

$$\|f\|_\infty = \sup\{|x| : x \text{ is in the essential range of } f\}$$

Example

If G is a countable group then $L^\infty(G) = \mathcal{L}^\infty(G)$. That is, $L^\infty(G)$ is the Banach space of bounded functions $f : G \to \mathbb{R}$, with norm

$$\|f\|_\infty = \sup\{|f(g)| : g \in G\}$$

In particular if G is finite then $L^\infty(G) \cong \mathbb{R}^{|G|}$ is the Banach space of all functions $f : G \to \mathbb{R}$ with norm

$$\|f\|_\infty = \max\{|f(g)| : g \in G\}$$

Action of G on $L^\infty(G)$ and characteristic functions

The group G acts on $L^\infty(G)$ on the left: for all $x \in G$, $f \in L^\infty(G)$ and $g \in G$

$$(g \cdot f)(x) = f(g^{-1}x)$$

This is called the left-regular representation.

For any subset $A \subseteq G$, define the characteristic function $\chi_A : G \to \mathbb{R}$ by

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

Note that $\chi_A \in L^\infty(G)$, and that if A and B are disjoint then $\chi_{A \cup B} = \chi_A + \chi_B$.

For any $g \in G$

$$(g \cdot \chi_A)(x) = \chi_A(g^{-1}x) = \chi_{gA}(x)$$

That is, $g \cdot \chi_A = \chi_{gA}$.

Bounded linear functionals

A linear functional on $L^\infty(G)$ is a linear transformation

$$\Lambda : L^\infty(G) \to \mathbb{R}$$

Given $f \in L^\infty(G)$, we say that $f \geq 0$ if $f(x) \geq 0$ for almost all $x \in G$.

Example

$\chi_A \geq 0$ for each measurable $A \subseteq G$

Invariant means and amenability

Definition

Let G be a locally compact group. An invariant mean is a linear functional $m : L^\infty(G) \to \mathbb{R}$ such that:

1. $m(f) \geq 0$ if $f \geq 0$
2. $m(\chi_G) = 1$
3. $m(g \cdot f) = m(f)$ for all $g \in G$ and $f \in L^\infty(G)$

Definition

A locally compact group G is amenable if it admits an invariant mean.

Invariant means for discrete groups

Let G be a discrete group, so that $L^\infty(G) = \mathcal{L}^\infty(G)$ is the space of all bounded functions on G.

Suppose that G is amenable with invariant mean $m : L^\infty(G) \to \mathbb{R}$.

We define a measure μ on *all* subsets of G as follows. For each $A \subseteq G$, put

$$\mu(A) := m(\chi_A)$$

Then $\mu(A) \geq 0$ for all $A \subseteq G$, since $m(f) \geq 0$ if $f \geq 0$.

To see that μ is finitely additive, suppose $A, B \subseteq G$ with $A \cap B = \emptyset$. Then

$$\begin{aligned}
\mu(A \cup B) &= m(\chi_{A\cup B}) \\
&= m(\chi_A + \chi_B) \text{ since } A \text{ and } B \text{ are disjoint} \\
&= m(\chi_A) + m(\chi_B) \text{ since } m \text{ is linear} \\
&= \mu(A) + \mu(B)
\end{aligned}$$

Invariant means for discrete groups

From the properties

1. $m(\chi_G) = 1$
2. $m(g \cdot f) = m(f)$ for all $g \in G$ and $f \in L^\infty(G)$

we obtain respectively that

$$\mu(G) = m(\chi_G) = 1$$

and that for each $A \subseteq G$ and $g \in G$

$$\mu(gA) = m(\chi_{gA}) = m(g \cdot \chi_A) = m(\chi_A) = \mu(A)$$

so μ is G–invariant.

That is, if a discrete group G has an invariant mean m, we obtain a finitely additive G–invariant measure μ defined on all subsets of G, such that $\mu(G) = 1$.

Discrete paradoxical groups are not amenable

Combining this with the easy direction of Tarski's Theorem:

Theorem (Tarski)

Suppose a group G acts on a set X. Then there is a finitely additive, G–invariant measure μ on X such that $\mu(X) = 1$ and μ is defined on all subsets of X if and only if X is not G–paradoxical.

we obtain:

Corollary

Let G be a discrete group. If G is paradoxical then G is not amenable.

In particular:

Corollary

The free group of rank 2, with the discrete topology, is not amenable. If any discrete group G has a free nonabelian subgroup, then G is not amenable.

Finite groups are amenable

Let G be a finite group. Then $L^\infty(G) = \mathcal{L}^\infty(G)$ is the space of all functions $f : G \to \mathbb{R}$.

We construct an invariant mean $m : L^\infty(G) \to \mathbb{R}$ by averaging. Given $f : G \to \mathbb{R}$ define

$$m(f) := \frac{1}{|G|} \sum_{x \in G} f(x)$$

Then it is not hard to verify that m is a linear functional such that

1. $m(f) \geq 0$ if $f \geq 0$
2. $m(\chi_G) = 1$
3. $m(g \cdot f) = m(f)$ for all $g \in G$ and $f \in L^\infty(G)$

Theorem

Let G be a finite group. Then G is amenable.