The Banach–Tarski Paradox and Amenability
Lecture 14: Quasi-Isometries

6 September 2012

Dependence of Cayley graph on generating sets

Let G be a group with finite generating sets $S \neq S'$. We assume that $S = S^{-1}$ and $S' = (S')^{-1}$, and that $1_G \notin S$ and $1_G \notin S'$.

Examples

1. $G = D_{2n}$ with $S = \{r, r^{-1}, s\}$ and $S' = \{s, t\}$.
2. $G = \mathbb{Z}$ with $S = \{\pm 1\}$ and $S' = \{\pm 2, \pm 3\}$.
3. $G = \mathbb{Z}^2$ with $S = \{(\pm 1, 0), (0, \pm 1)\}$ and $S' = \{(1, 2), (-1, -2), (2, -3), (-2, 3)\}$.

What is the relationship between the Cayley graphs $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, S')$?

We will show that these graphs are quasi-isometric, which means that, roughly speaking, $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, S')$ "look the same from far away".

Word length

Let G be a group with finite generating set S. We assume that $S = S^{-1}$, where $S^{-1} = \{s^{-1} \mid s \in S\}$, and that $1_G \notin S$.

The word length of an element $g \in G$ with respect to S

$$\ell_S(g) := \min\{\text{length}\, w \mid w \text{ a reduced word on } S, w \equiv g\}$$

That is, $\ell_S(g)$ is the length of a minimal reduced word for g.

Examples

1. In $C_7 = \langle r \mid r^7 = e \rangle$ with $S = \{r, r^{-1}\}$, we have

$$\ell_S(e) = 0, \quad \ell_S(r) = \ell_S(r^6) = 1$$

$$\ell_S(r^2) = \ell_S(r^5) = 2, \quad \ell_S(r^3) = \ell_S(r^4) = 3$$

Not all reduced words for a given $g \in G$ will be minimal.

More examples of word length

1. The length function depends on the generating set. In D_{2n} with generators $S = \{r, r^{-1}, s\}$ the element r has length $\ell_S(r) = 1$. But with generators $S' = \{s, t\}$ the element $r = st$ has length $\ell_{S'}(st) = 2$.
2. There may be more than one minimal reduced word for a given group element. For example in $D_6 = \langle s, t \mid s^2 = t^2 = 1, (st)^3 = 1 \rangle$, the words *sts* and *tst* are both reduced, both evaluate to the same group element and both are of minimal length.
3. In $\mathbb{Z}^2$ with generating set $S = \{(\pm 1, 0), (0, \pm 1)\}$, if $g = (m, n)$ then we have $\ell_S(g) = |m| + |n|$.
4. In a free group with a free generating set, there is a unique minimal reduced word for each group element.

Word metric on a finitely generated group

The word metric on G with respect to S, denoted d_S, is given by

$$d_S(g_1, g_2) = \ell_S(g_1^{-1}g_2) = \min\{\text{length } w \mid w \text{ reduced on } S, w \equiv g_1^{-1}g_2\}$$

Lemma

d_S is a metric.

Proof.

This follows from properties of ℓ_S:

- For all $g \in G$, $\ell_S(g) \geq 0$ with $\ell_S(g) = 0$ if and only if $g = e$.
- For all $g \in G$, $\ell_S(g) = \ell_S(g^{-1})$.
- For all $g, h \in G$, $\ell_S(gh) \leq \ell_S(g) + \ell_S(h)$.

□

Word metric on a finitely generated group

Lemma

The left-action of G on itself is an action by isometries with respect to the word metric.

Proof.

Any action is by bijections. This action is distance-preserving since for all $g, g_1, g_2 \in G$

$$\begin{aligned} d_S(gg_1, gg_2) &= \ell_S((gg_1)^{-1}gg_2) \\ &= \ell_S(g_1^{-1}g^{-1}gg_2) \\ &= \ell_S(g_1^{-1}g_2) \\ &= d_S(g_1, g_2) \end{aligned}$$

□

Word metric on Cayley graphs

The word metric on G with respect to S, denoted d_S, is given by

$$d_S(g_1, g_2) = \ell_S(g_1^{-1}g_2) = \min\{\text{length}\, w \mid w \text{ reduced on } S, w \equiv g_1^{-1}g_2\}$$

The Cayley graph of G with respect to S, denoted $\text{Cay}(G, S)$, is the graph with

- vertex set the elements of G
- an oriented edges (g, gs) for each $g \in G$ and $s \in S$

Equip the Cayley graph with the path metric by assigning length 1 to each edge, and defining the distance between two points to be the length of the shortest path which connects them. Then for each $g \in G$, the word length $\ell_S(g)$ is the length of a shortest path connecting g with the identity e.

The path metric on $\text{Cay}(G, S)$, restricted to the set of vertices, agrees with the word metric d_S on G. So we can talk about the word metric on the Cayley graph. The left-action of G on $\text{Cay}(G, S)$ is then an action by isometries.

Isometries and bi-Lipschitz maps

Let (X, d_X) and (Y, d_Y) be metric spaces.

Definitions

1. A function $f : X \to Y$ is an isometric embedding if for all $x_1, x_2 \in X$,
$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$$
An isometric embedding f is an isometry if in addition, f is surjective.
2. A function $f : X \to Y$ is a bi-Lipschitz embedding if there is a constant $\lambda > 0$ such that for all $x_1, x_2 \in X$,
$$\frac{1}{\lambda} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2)$$
A bi-Lipschitz embedding f is a bi-Lipschitz equivalence if in addition, f is surjective.

Bi-Lipschitz equivalence of word metrics

Specialise to the case $X = Y$ and let d and d' be two metrics on X. We say that (X, d) is bi-Lipschitz equivalent to (X, d') if there is a constant $\lambda > 0$ such that for all $x_1, x_2 \in X$,

$$\frac{1}{\lambda} d(x_1, x_2) \leq d'(x_1, x_2) \leq \lambda d(x_1, x_2)$$

In other words, the identity map on X is a bi-Lipschitz equivalence from (X, d) to (X, d').

Proposition

Let G be a group with finite generating sets $S \neq S'$. We assume that $S = S^{-1}$ and $S' = (S')^{-1}$, and that $1_G \notin S$ and $1_G \notin S'$. Then G with the word metric d_S is bi-Lipschitz equivalent to G with the word metric $d_{S'}$.

Word metrics are bi-Lipschitz equivalent

We want to show that there is a constant $\lambda > 0$ such that for any $g, h \in G$,

$$\frac{1}{\lambda} d_S(g,h) \leq d_{S'}(g,h) \leq \lambda d_S(g,h)$$

By definition of the word metric, it is enough to show that for all $g \in G$,

$$\frac{1}{\lambda} \ell_S(g) \leq \ell_{S'}(g) \leq \lambda \ell_S(g)$$

Let $S = \{s_1, \ldots, s_n\}$ and $S' = \{s'_1, \ldots, s'_m\}$. Let

$$C = \max\{\ell_{S'}(s_i) \mid 1 \leq i \leq n\}$$

$$C' = \max\{\ell_S(s'_j) \mid 1 \leq j \leq m\}$$

Word metrics are bi-Lipschitz equivalent

Let $g \in G$. Suppose $s_{i_1} s_{i_2} \cdots s_{i_n}$ is a minimal reduced word for g. Then in particular, $\ell_S(g) = n$.
We have

$$\begin{aligned}
\ell_{S'}(g) &= \ell_{S'}(s_{i_1} s_{i_2} \cdots s_{i_n}) \\
&\leq \ell_{S'}(s_{i_1}) + \ell_{S'}(s_{i_2}) + \cdots + \ell_{S'}(s_{i_n}) \\
&\leq Cn \\
&= C\ell_S(g)
\end{aligned}$$

Similarly, for all $g \in G$, $\ell_S(g) \leq C'\ell_{S'}(g)$. Combining these inequalities and choosing $\lambda = \max(C, C')$, we obtain the result.

We have shown that the word metric on a finitely generated group G depends on the (finite) generating set only up to bi-Lipschitz equivalence.

Quasi-isometries

Let (X, d_X) and (Y, d_Y) be metric spaces.

Definition

A function $f : X \to Y$ is a quasi-isometric embedding if there exist constants $\lambda, C > 0$ such that for all $x_1, x_2 \in X$,

$$\frac{1}{\lambda} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + C$$

An quasi-isometric embedding f is a quasi-isometry if in addition, there is a constant $D > 0$ so that every point of Y is within distance D of some point of the image $f(X)$.

A quasi-isometry f as above is sometimes called a (λ, C)–quasi-isometry.

If there is a quasi-isometry $X \to Y$, we say that X and Y are quasi-isometric. Quasi-isometry is an equivalence relation on the set of metric spaces.

Examples of quasi-isometries

1. (G, d_S) is $(\lambda, 0)$–quasi-isometric to $(G, d_{S'})$ for any finitely generated group G and finite generating sets S and S', since (G, d_S) and $(G, d_{S'})$ are bi-Lipschitz equivalent. (The constant λ will depend on G, S and S'.)
2. Let $f : \mathbb{R} \to \mathbb{Z}$ be the floor function. Then f is a quasi-isometry. Quasi-isometries don't have to be injective or continuous. We just don't care what happens at small scales.
3. Similarly we can define a quasi-isometry $\mathbb{R}^2 \to \mathbb{Z}^2$. This encodes the idea that the lattice of points in $\mathbb{Z}^2$ "looks the same as $\mathbb{R}^2$ from far away".

Cayley graphs are well-defined up to quasi-isometry

Proposition

Let G be a group with finite generating sets $S \neq S'$. We assume that $S = S^{-1}$ and $S' = (S')^{-1}$, and that $1_G \notin S$ and $1_G \notin S'$. Then $\mathrm{Cay}(G, S)$ is quasi-isometric to $\mathrm{Cay}(G, S')$.

That is, the Cayley graph for a finitely generated group G depends on the choice of (finite) generating set only up to quasi-isometry.

Proof.

Shrink the edges onto adjacent vertices and then use the fact that (G, d_S) is bi-Lipschitz equivalent to $(G, d_{S'})$. □

Corollary

A finitely generated group G is quasi-isometric to any of its Cayley graphs.

More examples of quasi-isometries

1. Any two metric spaces of finite diameter are quasi-isometric.
2. In particular, let (X, d) be a metric space of finite diameter. Then X is quasi-isometric to a point.
3. The Cayley graph of a finite group is a finite graph. So every finite group is quasi-isometric to every other finite group, and every finite group is quasi-isometric to the trivial group.

Question (Gromov)

Classify infinite finitely generated groups up to quasi-isometry.

Bi-Lipschitz equivalence is not the same as quasi-isometric equivalence

Theorem (Dymarz 2010)

There are finitely generated groups that are quasi-isometric to each other but are not bi-Lipschitz equivalent.

The examples given by Dymarz are lamplighter groups, which are amenable.

Theorem (Whyte 1999)

Let $f : G \to H$ be a quasi-isometry of nonamenable finitely generated groups. Then there a bi-Lipschitz equivalence $f' : G \to H$ and a constant $K > 0$ such that for all $g \in G$,

$$d_H(f(g), f'(g)) \leq K.$$

That is, every quasi-isometry is a uniformly bounded distance from a bi-Lipschitz equivalence.