The Banach–Tarski Paradox and Amenability
Lecture 15: Følner Conditions

15 September 2011

Cayley graphs are well-defined up to quasi-isometry

Proposition

Let G be a group with finite generating sets $S \neq S'$. Then $\mathrm{Cay}(G, S)$ *is quasi-isometric to* $\mathrm{Cay}(G, S')$*.*

That is, the Cayley graph for a finitely generated group G depends on the choice of (finite) generating set only up to quasi-isometry.

Corollary

A finitely generated group G is quasi-isometric to any of its Cayley graphs.

Motivation for the Følner condition

Proposition

Let G be a discrete group. Then there is an invariant mean $m : L^\infty(G) \to \mathbb{R}$ if and only if there is a finitely additive G–invariant measure μ defined on all subsets of G, such that $\mu(G) = 1$.

We already showed that given m, we can obtain μ by defining

$$\mu(A) := m(\chi_A)$$

for all subsets $A \subset G$. For the converse, if $s : G \to \mathbb{R}$ is a simple function with

$$s = \sum_{i=1}^{n} c_i \chi_{A_i}$$

for pairwise disjoint subsets A_i of G and $c_i \in \mathbb{R}$, we define

$$m(s) = \sum_{i=1}^{n} c_i \mu(A_i)$$

Motivation for the Følner condition

For bounded $f : G \to \mathbb{R}$ with $f \geq 0$ we put

$$m(f) := \sup\{m(s) : s \text{ is simple, } 0 \leq s \leq f\}$$

and for general bounded $f : G \to \mathbb{R}$ we put

$$m(f) := m(f^+) - m(f^-)$$

Check that m is linear and that

1. $m(f) \geq 0$ if $f \geq 0$
2. $m(\chi_G) = 1$
3. $m(g \cdot f) = m(f)$ for all $g \in G$ and $f \in L^\infty(G)$

Where did we use $\mu(G) = 1$?

Boundaries of vertex subsets

Let G be a finitely generated group with Cayley graph $\mathrm{Cay}(G, S)$.

Let U be a nonempty subset of the set of vertices of $\mathrm{Cay}(G, S)$. Then the boundary of U, denoted ∂U, is the set of vertices of $\mathrm{Cay}(G, S)$ which are adjacent to a vertex of U but are not in U. Equivalently,

$$\partial U = U^c \cap \{g \in G \mid g = us \text{ for some } u \in U,\ s \in S\}$$

The Følner condition for finitely generated groups

Let G be a finitely generated group with Cayley graph $\mathrm{Cay}(G, S)$.

Definition

A finitely generated group G satisfies the Følner condition if for every $\varepsilon > 0$, there is a nonempty finite subset $U \subseteq G$ such that

$$\frac{|\partial U|}{|U|} < \varepsilon$$

That is, for every $\varepsilon > 0$ there is a nonempty finite set of vertices U of the Cayley graph $\mathrm{Cay}(G, S)$ such that $|\partial U| < \varepsilon |U|$.

Lemma

This definition is independent of the choice of (finite) generating set for G.

A set U as in the definition above is called a Følner set.

Følner sequences for finitely generated groups

Let G be a finitely generated group with Cayley graph $\mathrm{Cay}(G, S)$.

Definition

A Følner sequence in a finitely generated group G is a sequence $F_1, F_2, \ldots$ of finite subsets of G such that

$$\lim_{j\to\infty} \frac{|\partial F_j|}{|F_j|} = 0$$

Lemma

This definition is independent of the choice of (finite) generating set for G.

A finitely generated group G satisfies the Følner condition if and only if it has a Følner sequence.

Examples

1. Any finite group satisfies the Følner condition. Take U to be the whole group, then $|\partial U| = 0$.
2. $\mathbb{Z}$ has a Følner sequence. Consider the generating set $S = \{\pm 1\}$ and let F_j be the set of integers in the interval $[-j, j]$. Then $|\partial F_j| = 2$ while $|F_j| = 2j + 1$, so
$$\lim_{j \to \infty} \frac{|\partial F_j|}{|F_j|} = 0$$
3. $\mathbb{Z}^n$ has a Følner sequence. Take U_j to be the cube $[-j, j]^n$.
4. We will show in just a moment that the free group of rank 2 does not have a Følner sequence.

Relationship to amenability

We will prove:

Theorem (Følner 1950s)

Let G be a finitely generated group. Then G satisfies the Følner condition if and only if G is amenable (as a discrete group).

Corollary

There is a a finitely additive translation-invariant measure μ defined on all subsets of $\mathbb{Z}$, such that $\mu(\mathbb{Z}) = 1$.

By finite-additivity and translation-invariance, if A is any finite subset of $\mathbb{Z}$ then $\mu(A) = 0$, and if $A = n\mathbb{Z}$ then $\mu(A) = \frac{1}{n}$. But there are other infinite subsets A of $\mathbb{Z}$ such that $1 > \mu(A) > 0$ and A is not constructible.

Motivation for the Følner condition

Suppose that a finitely generated group G has a Følner sequence $\{F_j\}_{j=1}^{\infty}$ such that $G = \cup_{j=1}^{\infty} F_j$. We might try to construct a finitely additive G–invariant measure μ defined on all subsets of G, such that $\mu(G) = 1$, by putting

$$\mu(A) := \lim_{j \to \infty} \frac{|A \cap F_j|}{|F_j|}$$

for all subsets $A \subset G$.
BUT this limit does not in general exist. One fix is to use something called a non-principal ultrafilter, the existence of which is equivalent to the Axiom of Choice.

Relationship to quasi-isometries

Proposition

Suppose G and H are finitely generated groups such that G and H are quasi-isometric. Then G satisfies the Følner condition if and only if H satisfies the Følner condition.

Corollary

Suppose G and H are finitely generated groups such that G and H are quasi-isometric. Then G is amenable if and only if H is amenable.

That is, amenability is a quasi-isometry invariant (among finitely generated groups).

$F(a, b)$ does not satisfy the Følner condition

Proposition

Let $G = F(a, b)$ be the free group of rank 2. Then G does not have a Følner sequence.

It is enough to show that for all finite subsets U of $G = F(a, b)$

$$\frac{|\partial U|}{|U|} \geq 1$$

Let $S = \{a^{\pm 1}, b^{\pm 1}\}$ and let X_U be the subgraph of $\mathrm{Cay}(G, S)$ induced by U. That is, X_U has vertices U and edges (u, u') for all $u, u' \in U$ such that $u' \in \{ua^{\pm 1}, ub^{\pm 1}\}$.

We first show that for all finite subsets U such that X_U is *connected*,

$$\frac{|\partial U|}{|U|} \geq 2$$

$F(a, b)$ does not satisfy the Følner condition

Assume X_U is connected. For $i = 1, 2, 3, 4$ let v_i be the number of vertices of X_U of degree i. Let E be the number of edges of X_U. Then

$$E = v_1 + v_2 + v_3 + v_4 - 1$$

and also

$$E = \frac{v_1 + 2v_2 + 3v_3 + 4v_4}{2}$$

From this we obtain

$$|U| = v_1 + v_2 + v_3 + v_4 = \frac{3v_1 + 2v_2 + v_3 - 2}{2}$$

The boundary ∂U is the edges joined to vertices of degree $1, 2, 3$ only, so

$$|\partial U| = 3v_1 + 2v_2 + v_3$$

and a straightforward computation shows that

$$\frac{|\partial U|}{|U|} \geq 2$$

$F(a, b)$ does not satisfy the Følner condition

Now let $U_1, \ldots, U_k \subset U$ be such that $X_{U_1}, \ldots, X_{U_k}$ are the connected components of X_U. Then

$$\begin{aligned} \frac{|\partial U|}{|U|} &\geq \frac{|\partial U_1| + \cdots + |\partial U_k| - (k-1)}{|U|} \\ &\geq \frac{2|U_1| + \cdots + 2|U_k| - k}{|U|} \\ &\geq 2 - \frac{k}{|U|} \\ &\geq 1 \end{aligned}$$

since $|U| = |U_1| + \cdots + |U_k| \geq k$. Therefore the free group of rank 2 does not have a Følner sequence.

Reformulation of the Følner condition for finitely generated groups

Let G be a finitely generated group. For $U \subset G$, write $\partial_S U$ for the boundary of U in $\mathrm{Cay}(G, S)$. That is,

$$\partial_S U = U^c \cap \{g \in G \mid g = us \text{ for some } u \in U,\ s \in S\}$$

For any subset $K \subset G$, we now define

$$\partial_K U = U^c \cap \{g \in G \mid g = uk \text{ for some } u \in U,\ k \in K\}$$

Suppose that G satisfies the Følner condition. Let $\varepsilon > 0$ and let $K \subset G$ be a finite subset. Choose a finite generating set S which contains K. Then there exists a nonempty finite subset $U \subseteq G$ such that

$$|\partial_S U| < \varepsilon |U|$$

Since $K \subset S$, we have $\partial_K U \subset \partial_S U$ hence

$$|\partial_K U| < \varepsilon |U|$$

Reformulation of the Følner condition for finitely generated groups

Now suppose that for every $\varepsilon > 0$ and every finite subset $K \subset G$, there is a nonempty finite subset $U \subseteq G$ such that

$$|\partial_K U| < \varepsilon |U|$$

Then in particular for every finite generating set S of G, there is a nonempty finite subset $U \subseteq G$ such that

$$|\partial_S U| < \varepsilon |U|$$

We have shown that the Følner condition for finitely generated groups may be reformulated as:

Definition

A finitely generated group G satisfies the Følner condition if for every $\varepsilon > 0$ and every finite subset $K \subset G$, there is a nonempty finite subset $U \subseteq G$ such that

$$\frac{|\partial_K U|}{|U|} < \varepsilon$$

The Følner condition for discrete groups

Recall that for sets A and B, the symmetric difference of A and B is

$$A \triangle B := (A \cap B^c) \cup (B \cap A^c)$$

Lemma

Let G be a finitely generated group. Let $\varepsilon > 0$ and let $K \subset G$ be a finite subset of G. Then there is a nonempty finite subset $U \subseteq G$ such that

$$|\partial_K U| < \varepsilon |U|$$

if and only if there is a nonempty finite subset $V \subseteq G$ such that for all $x \in K$

$$|xV \triangle V| < \varepsilon |V|$$

The Følner condition for discrete groups

Definition

A discrete group G satisfies the Følner condition if for every $\varepsilon > 0$ and every finite subset $K \subset G$, there is a nonempty finite set $U \subseteq G$ such that for all $x \in K$

$$|xU \triangle U| < \varepsilon |U|$$

Lemma

If every finitely generated subgroup of a discrete group G satisfies the Følner condition, then G satisfies the Følner condition.

Proof.

Given $\varepsilon > 0$ and finite $K \subset G$, let H be the subgroup of G generated by K. Then we can take $U \subset H$. □

Theorem (Følner 1950s)

Let G be a discrete group. Then G satisfies the Følner condition if and only if G admits an invariant mean.

The Følner condition for locally compact groups

Let G be a locally compact group with Haar measure μ.

Definition

The locally compact group G satisfies the Følner condition if for every $\varepsilon > 0$ and every compact subset $K \subset G$, there is a Borel set $U \subseteq G$ of positive finite Haar measure $\mu(U)$ such that for all $x \in K$

$$\frac{\mu(xU \triangle U)}{\mu(U)} < \varepsilon$$

Example

Any compact group G satisfies the Følner condition. Take $U = G$.

Theorem (Greenleaf 1960s)

Let G be a locally compact group. Then G satisfies the Følner condition if and only if G admits an invariant mean.

We will prove this. It implies Følner's Theorem (the discrete case) above.