

The Banach–Tarski Paradox and Amenability

Lecture 19: Reiter's Property and the Følner Condition

9 October 2012

Invariant means and amenability

Definition

Let G be a locally compact group. An **invariant mean** is a linear functional $m : L^\infty(G) \rightarrow \mathbb{R}$ such that:

1. $m(f) \geq 0$ if $f \geq 0$
2. $m(\chi_G) = 1$
3. $m(g \cdot f) = m(f)$ for all $g \in G$ and $f \in L^\infty(G)$

Definition

A locally compact group G is **amenable** if it admits an invariant mean.

Equivalence of amenability and the Følner Condition

Last lecture we used the Hahn–Banach Theorem to prove:

G satisfies the Følner Condition $\implies G$ admits an invariant mean

Today we will define a property of locally compact groups called the Reiter Property. We'll then show (over ≥ 2 lectures):

G admits an invariant mean

$\implies G$ satisfies the Reiter Property

$\implies G$ satisfies the Følner Condition

and so obtain:

Theorem (Følner, Greenleaf)

Let G be a locally compact group. Then G admits an invariant mean if and only if G satisfies the Følner Condition.

Fubini's Theorem

Fubini's Theorem provides conditions under which the order of integration may be reversed.

A **complete** measure space is one in which all subsets of sets of measure zero are measurable (and have measure zero). The Lebesgue measure and counting measure are complete.

Theorem

Let (X, μ) and (Y, λ) be complete measure spaces. If $f : X \times Y \rightarrow \mathbb{R}$ is measurable (w.r.t. the product measure) then

$$\int_A \left(\int_B f(x, y) d\lambda \right) d\mu = \int_B \left(\int_A f(x, y) d\mu \right) d\lambda$$

for all measurable $A \subset X$, $B \subset Y$.

Fubini's Theorem

A σ -finite measure space is one which is the countable union of measurable sets of finite measure. Connected Lie groups are σ -finite under Haar measure.

Theorem

Let (X, μ) and (Y, λ) be σ -finite measure spaces. If either

$$\int_A \left(\int_B |f(x, y)| d\lambda \right) d\mu < \infty$$

or

$$\int_B \left(\int_A |f(x, y)| d\mu \right) d\lambda < \infty$$

then

$$\int_A \left(\int_B f(x, y) d\lambda \right) d\mu = \int_B \left(\int_A f(x, y) d\mu \right) d\lambda$$

The spaces $L^1(G)$ and $L^\infty(G)^*$

Let G be a locally compact group with Haar measure μ . Recall

$$\mathcal{L}^1(G) := \left\{ f : G \rightarrow \mathbb{R} \mid f \text{ is measurable and } \int_G |f| d\mu < \infty \right\}$$

$$L^1(G) := \mathcal{L}^1(G) / \{ \text{measurable functions } f = 0 \text{ } \mu\text{-a.e.} \}$$

Then $L^1(G)$ is a normed linear space with norm (by abuse of notation)

$$\|f\|_1 = \int_G |f| d\mu$$

$L^1(G)$ is the space of (equivalence classes of) integrable functions on G .

Recall the dual space

$$L^\infty(G)^* := \{ \text{bounded linear functionals on } L^\infty(G) \}$$

This is also a normed linear space, with the operator norm.

The spaces $L^1(G)$ and $L^\infty(G)^*$

Lemma

If $f \in L^1(G)$ and $\varphi \in L^\infty(G)$ then $f\varphi \in L^1(G)$.

Proof.

$f\varphi$ is measurable since f and φ are measurable, and

$$\int_G |f\varphi| d\mu \leq \|\varphi\|_\infty \int_G |f| d\mu < \infty$$

since f is integrable and φ bounded. □

The spaces $L^1(G)$ and $L^\infty(G)^*$

Corollary

There is an isometric embedding $L^1(G) \rightarrow L^\infty(G)^$, so we may identify $L^1(G)$ with a subspace of $L^\infty(G)^*$.*

Proof.

Let $f \in L^1(G)$. Define $\lambda_f : L^\infty(G) \rightarrow \mathbb{R}$ by

$$\lambda_f(\varphi) = \int_G f\varphi \, d\mu$$

Then λ_f is a linear functional on $L^\infty(G)$. We have

$$\|\lambda_f\| = \sup\{|\lambda_f(\varphi)| : \varphi \in L^\infty(G), \|\varphi\|_\infty = 1\} = \|f\|_1$$

so λ_f is bounded. Hence $\lambda_f \in L^\infty(G)^*$.

To see that $f \mapsto \lambda_f$ is an isometric embedding, let $f, f' \in L^1(G)$.

Then $\lambda_f - \lambda_{f'} = \lambda_{f-f'}$ so $\|\lambda_f - \lambda_{f'}\| = \|f - f'\|_1$ as required. $\square$

Means on $L^\infty(G)$

Definition

Let G be a locally compact group. An **mean** on $L^\infty(G)$ is a linear functional $m : L^\infty(G) \rightarrow \mathbb{R}$ such that:

1. $m(\varphi) \geq 0$ if $\varphi \geq 0$
2. $m(\chi_G) = 1$

Lemma

Let $\mathcal{M}$ be the set of all means on $L^\infty(G)$. Then $\mathcal{M}$ is a subset of the unit ball in $L^\infty(G)^$. That is, every mean is a bounded linear functional of norm at most 1.*

Proof.

Let $\varphi \in L^\infty(G)$. Then

$$-\|\varphi\|_\infty \chi_G \leq \varphi \leq \|\varphi\|_\infty \chi_G$$

It follows that $|m(\varphi)| \leq \|\varphi\|_\infty$, and so $\|m\| \leq 1$ as required. □

The set $L^1(G)_{1,+}$

We want a subset of $L^1(G)$ whose image in $L^\infty(G)^*$ consists of means. Define

$$L^1(G)_{1,+} = \{f \in L^1(G) : \|f\|_1 = 1 \text{ and } f \geq 0\}$$

Lemma

Every $f \in L^1(G)_{1,+}$ defines a mean $m_f : L^\infty(G) \rightarrow \mathbb{R}$ via:

$$m_f(\varphi) := \int_G f \varphi \, d\mu$$

Lemma

The image of $L^1(G)_{1,+}$ is weak* dense in $\mathcal{M}$.

Proof.

Let $m \in \mathcal{M}$. By definition of the weak* topology on $L^\infty(G)^*$, we need to show that there is a net $\{f_n\}$ in $L^1(G)_{1,+}$ so that $m_{f_n}(\varphi) \rightarrow m(\varphi)$ for all $\varphi \in L^\infty(G)$. This follows from the Hahn–Banach Theorem.

Reiter's Property

The group G acts on $L^1(G)_{1,+}$ via the left regular representation

$$g \cdot f(x) = f(g^{-1}x) \quad \forall g \in G, f \in L^1(G)_{1,+} \text{ and } x \in G$$

Definition

A locally compact group G satisfies **Reiter's Property** if for every $\varepsilon > 0$ and every compact subset K of G , there is an $f \in L^1(G)_{1,+}$ such that for all $k \in K$

$$\|k \cdot f - f\|_1 \leq \varepsilon$$

Today we'll show that Reiter's Property implies the Følner Condition. Then in the next lecture(s) we'll prove:

Theorem (Hulanicki 1965, Reiter 1966)

Let G be a locally compact group. If G admits an invariant mean then G satisfies Reiter's Property.

Reiter's Property implies the Følner Condition

Lemma

Let G be a locally compact group with Haar measure μ . Let $f, f' \in L^1(G)$ with $f, f' \geq 0$. For every $t \geq 0$ let

$$E_t = \{x \in G : f(x) \geq t\}$$

and

$$E'_t = \{x \in G : f'(x) \geq t\}$$

Then

$$\|f - f'\|_1 = \int_0^\infty \mu(E_t \triangle E'_t) dt$$

In particular, putting $f' = 0$

$$\|f\|_1 = \int_0^\infty \mu(E_t) dt$$

Proof of Lemma

Write χ_t for the characteristic function of $[t, \infty)$. Then for $s, s' \in [0, \infty)$

$$\int_0^\infty |\chi_t(s) - \chi_t(s')| dt = |s - s'|$$

so for all $x \in G$

$$\int_0^\infty |\chi_t(f(x)) - \chi_t(f'(x))| dt = |f(x) - f'(x)|$$

Hence by Fubini's Theorem

$$\begin{aligned} \int_0^\infty \|\chi_t \circ f - \chi_t \circ f'\|_1 dt &= \int_0^\infty \int_G |\chi_t(f(x)) - \chi_t(f'(x))| d\mu(x) dt \\ &= \int_G |f(x) - f'(x)| d\mu(x) \\ &= \|f - f'\|_1 \end{aligned}$$

Proof of Lemma

Now

$$\chi_t \circ f(x) = \begin{cases} 1 & \text{if } f(x) \geq t \\ 0 & \text{if } f(x) < t \end{cases}$$

Thus $|\chi_t \circ f - \chi_t \circ f'|$ is the characteristic function of $E_t \triangle E'_t$. So

$$\|\chi_t \circ f - \chi_t \circ f'\|_1 = \mu(E_t \triangle E'_t)$$

and thus

$$\|f - f'\|_1 = \int_0^\infty \mu(E_t \triangle E'_t) dt$$

as required.

Reiter's Property implies the Følner Condition

Assume that the locally compact group G satisfies Reiter's Property. Let $\varepsilon > 0$ and let Q be a compact subset of G containing e . Then

$$K := Q^2 = \{qq' : q, q' \in Q\}$$

is a compact subset of G . So by Reiter's Property, there exists $f \in L^1(G)_{1,+}$ such that for all $k \in K$

$$\|k \cdot f - f\|_1 \leq \frac{\varepsilon \mu(Q)}{2\mu(K)}$$

Hence

$$\int_K \|k \cdot f - f\|_1 dk \leq \frac{\varepsilon \mu(Q)}{2}$$

Reiter's Property implies the Følner Condition

For $t \geq 0$ let E_t be as in the Lemma. Then for each $k \in K$,

$$\begin{aligned} kE_t &= \{kx \in G : f(x) \geq t\} \\ &= \{y \in G : f(k^{-1}y) \geq t\} \\ &= \{y \in G : (k \cdot f)(y) \geq t\} \end{aligned}$$

Thus by the Lemma, for every $k \in K$

$$\|k \cdot f - f\|_1 = \int_0^\infty \mu(kE_t \triangle E_t) dt$$

Hence

$$\int_0^\infty \mu(E_t) \left(\int_K \frac{\mu(kE_t \triangle E_t)}{\mu(E_t)} dk \right) dt = \int_K \|k \cdot f - f\|_1 dk \leq \frac{\varepsilon \mu(Q)}{2}$$

Reiter's Property implies the Følner Condition

We have

$$\int_0^\infty \mu(E_t) \left(\int_K \frac{\mu(kE_t \triangle E_t)}{\mu(E_t)} dk \right) dt \leq \frac{\varepsilon \mu(Q)}{2}$$

By the Lemma,

$$\int_0^\infty \mu(E_t) dt = \|f\|_1 = 1$$

so it follows that there is some $t \geq 0$ so that $0 < \mu(E_t) < \infty$ and

$$\int_K \frac{\mu(kE_t \triangle E_t)}{\mu(E_t)} dk \leq \frac{\varepsilon \mu(Q)}{2}$$

We will show that this set E_t is a Følner set, for $2\varepsilon > 0$ and Q compact. Note that it's enough to consider compact sets containing e .

Reiter's Property implies the Følner Condition

We have chosen $t \geq 0$ such that $0 < \mu(E_t) < \infty$ and

$$\int_K \frac{\mu(kE_t \triangle E_t)}{\mu(E_t)} dk \leq \frac{\varepsilon \mu(Q)}{2}$$

Define

$$A = \left\{ k \in K : \frac{\mu(kE_t \triangle E_t)}{\mu(E_t)} \leq \varepsilon \right\}$$

We claim that $Q \subset AA^{-1}$. Assuming this, let $a_1, a_2 \in A$. Then

$$(a_1 a_2^{-1} E_t \triangle E_t) \subset ((a_1 a_2^{-1} E_t \triangle a_1 E_t) \cup (a_1 E_t \triangle E_t))$$

so as μ is G -invariant

$$\begin{aligned} \mu(a_1 a_2^{-1} E_t \triangle E_t) &\leq \mu(a_2^{-1} E_t \triangle E_t) + \mu(a_1 E_t \triangle E_t) \\ &= \mu(a_2 E_t \triangle E_t) + \mu(a_1 E_t \triangle E_t) \\ &\leq 2\varepsilon \mu(E_t) \end{aligned}$$

Thus for all $q \in Q$

$$\mu(qE_t \triangle E_t) \leq 2\varepsilon \mu(E_t)$$

Reiter's Property implies the Følner Condition

We chose $t \geq 0$ such that $0 < \mu(E_t) < \infty$ and

$$\int_K \frac{\mu(kE_t \Delta E_t)}{\mu(E_t)} dk \leq \frac{\varepsilon \mu(Q)}{2}$$

and defined

$$A = \left\{ k \in K : \frac{\mu(kE_t \Delta E_t)}{\mu(E_t)} \leq \varepsilon \right\}$$

Then $\mu(K \setminus A) < \frac{\mu(Q)}{2}$.

We use this to prove the claim that $Q \subset AA^{-1}$. Let $q \in Q$. It is enough to show that $\mu(qA \cap A) > 0$. Now since $K = Q^2$ and Q contains e , from set inclusions we get

$$\mu(Q) = \mu(qQ) \leq \mu(qK \cap K) \leq \mu(qA \cap A) + \mu(K \setminus A) + \mu(q(K \setminus A))$$

The right-hand side equals

$$\mu(qA \cap A) + 2\mu(K \setminus A) < \mu(qA \cap A) + \mu(Q)$$

thus $\mu(qA \cap A) > 0$, as required. This completes the proof.