

The Banach–Tarski Paradox and Amenability

Lecture 20: Invariant Mean implies Reiter's Property

11 October 2012

Invariant means and amenability

Definition

Let G be a locally compact group. An **invariant mean** is a linear functional $m : L^\infty(G) \rightarrow \mathbb{R}$ such that:

1. $m(f) \geq 0$ if $f \geq 0$
2. $m(\chi_G) = 1$
3. $m(g \cdot f) = m(f)$ for all $g \in G$ and $f \in L^\infty(G)$

Definition

A locally compact group G is **amenable** if it admits an invariant mean.

Invariant mean implies Følner

Today we will finish proving:

Theorem (Følner, Greenleaf)

Let G be a locally compact group. Then G admits an invariant mean if and only if G satisfies the Følner Condition.

In Lecture 18 we showed:

G satisfies the Følner Condition $\implies G$ admits an invariant mean
and in Lecture 19 we defined a property of locally compact groups called the Reiter Property and showed:

G satisfies the Reiter Property $\implies G$ satisfies the Følner Condition

So today we will show (most of)

G admits an invariant mean $\implies G$ satisfies the Reiter Property

The spaces $L^p(G)$ and $L^\infty(G)^*$

Let G be a locally compact group with Haar measure μ . Let $1 \leq p < \infty$. Recall

$$\mathcal{L}^p(G) := \left\{ f : G \rightarrow \mathbb{R} \mid f \text{ is measurable and } \int_G |f|^p d\mu < \infty \right\}$$

$$L^p(G) := \mathcal{L}^p(G) / \{ \text{measurable functions } f = 0 \text{ } \mu\text{-a.e.} \}$$

Then $L^p(G)$ is a normed linear space with norm (by abuse of notation)

$$\|f\|_p = \left(\int_G |f|^p d\mu \right)^{1/p}$$

In particular, $L^1(G)$ is the space of (equivalence classes of) integrable functions on G .

Recall the dual space

$$L^\infty(G)^* := \{ \text{bounded linear functionals on } L^\infty(G) \}$$

This is also a normed linear space, with the operator norm.

Continuity of the left regular representation

For $1 \leq p \leq \infty$ the group G acts on $L^p(G)$ via

$$g \cdot f(x) = f(g^{-1}x) \quad \forall g \in G, f \in L^p(G) \text{ and } x \in G$$

It is an exercise to show:

Lemma

Let $f \in L^p(G)$. The mapping $g \mapsto g \cdot f$ from $G \rightarrow L^p(G)$ is continuous if $1 \leq p < \infty$. This mapping is not necessarily continuous for $p = \infty$.

Definition

Let $UCB(G)$ be the subspace of $L^\infty(G)$ given by

$$UCB(G) = \{f \in L^\infty(G) \mid g \mapsto g \cdot f \text{ is continuous}\}$$

The G -action on $L^\infty(G)$ preserves $UCB(G)$.

If G is compact or discrete, then $L^\infty(G) = UCB(G)$.

Means on $L^\infty(G)$ and $UCB(G)$

Definition

Let G be a locally compact group. An **mean** on $L^\infty(G)$ is a linear functional $m : L^\infty(G) \rightarrow \mathbb{R}$ such that:

1. $m(\varphi) \geq 0$ if $\varphi \geq 0$
2. $m(\chi_G) = 1$

Similarly we can define a mean on $UCB(G)$.

Means on $L^\infty(G)$ and $L^1(G)$

We proved in Lecture 19:

Lemma

Let $\mathcal{M}$ be the set of all means on $L^\infty(G)$. Then $\mathcal{M}$ is a subset of the unit ball in $L^\infty(G)^$. That is, a mean is a bounded linear functional of norm at most 1.*

Lemma

If $f \in L^1(G)$ and $\varphi \in L^\infty(G)$ then $f\varphi \in L^1(G)$.

Corollary

There is an isometric embedding $L^1(G) \rightarrow L^\infty(G)^$, so we may regard $L^1(G)$ as a subspace of $L^\infty(G)^*$.*

The embedding is $f \mapsto \lambda_f$ where for $\varphi \in L^\infty(G)$

$$\lambda_f(\varphi) = \int_G f\varphi \, d\mu$$

The set $L^1(G)_{1,+}$

In Lecture 19 we also defined a subset of $L^1(G)$ whose image in $L^\infty(G)$ consists of means by

$$L^1(G)_{1,+} = \{f \in L^1(G) : \|f\|_1 = 1 \text{ and } f \geq 0\}$$

Lemma

Every $f \in L^1(G)_{1,+}$ defines a mean $m_f : L^\infty(G) \rightarrow \mathbb{R}$ via:

$$m_f(\varphi) := \int_G f \varphi \, d\mu$$

Density of $L^1(G)_{1,+}$ in $\mathcal{M}$

Lemma

The image of $L^1(G)_{1,+}$ is weak dense in $\mathcal{M}$.*

Let $m \in \mathcal{M} \subset L^\infty(G)^*$. By definition of the weak* topology on $L^\infty(G)^*$, we need to show that there is a sequence $\{f_i\}_i$ in $L^1(G)_{1,+}$ so that for all $\varphi \in L^\infty(G)$

$$m_{f_i}(\varphi) \rightarrow m(\varphi)$$

Use:

Theorem (Hahn–Banach)

Let M be a linear subspace of a Banach space X and let $x_0 \in X$. Then x_0 is in $\overline{M}$ (the closure of M) if and only if there does not exist $f \in X^$ such that $f(x) = 0$ for all $x \in M$ but $f(x_0) \neq 0$.*

Reiter's Property

The group G acts on $L^1(G)_{1,+}$ via

$$g \cdot f(x) = f(g^{-1}x) \quad \forall g \in G, f \in L^1(G)_{1,+} \text{ and } x \in G$$

Definition

A locally compact group G satisfies **Reiter's Property** if for every $\varepsilon > 0$ and every compact subset K of G , there is an $f \in L^1(G)_{1,+}$ such that for all $k \in K$

$$\|k \cdot f - f\|_1 \leq \varepsilon$$

Today we'll prove:

Theorem (Hulanicki 1965, Reiter 1966)

Let G be a locally compact group. If G admits an invariant mean then G satisfies Reiter's Property.

using the proof in Bekka–de la Harpe–Valette.

Properties of the set $L^1(G)_{1,+}$

Lemma

$L^1(G)_{1,+}$ is a **convex** subset of $L^1(G)$.

Proof.

If $f, g \in L^1(G)_{1,+}$ and $t \in [0, 1]$ then $tf + (1 - t)g \geq 0$ and

$$\begin{aligned}\|tf + (1 - t)g\|_1 &= \int_G |tf + (1 - t)g| d\mu \\ &= \int_G tf + (1 - t)g d\mu \\ &= t \int_G f d\mu + (1 - t) \int_G g d\mu \\ &= t\|f\|_1 + (1 - t)\|g\|_1 \\ &= 1\end{aligned}$$

so $tf + (1 - t)g \in L^1(G)_{1,+}$ as required for convexity. □

Properties of the set $L^1(G)_{1,+}$

Lemma

$L^1(G)_{1,+}$ is closed under *convolution*, where

$$(f * g)(x) := \int_G f(y)g(y^{-1}x) d\mu(y)$$

for all $f, g \in L^1(G)_{1,+}$ and $x \in G$.

Proof.

We have $f * g \geq 0$. By Fubini's Theorem, since f and g are integrable

$$\int_G f * g d\mu = \left(\int_G f d\mu \right) \left(\int_G g d\mu \right)$$

hence

$$\|f * g\|_1 = \|f\|_1 \|g\|_1 = 1$$



Properties of the set $L^1(G)_{1,+}$

Definition

Let $UCB(G)$ be the subspace of $L^\infty(G)$ given by

$$UCB(G) = \{f \in L^\infty(G) \mid g \mapsto g \cdot f \text{ is continuous}\}$$

Lemma

If $f \in L^1(G)_{1,+}$ and $\varphi \in L^\infty(G)$ then

$$f * \varphi = \int_G f(y)(y \cdot \varphi) d\mu(y)$$

is in $UCB(G)$.

Proof.

Exercise.



Invariant mean implies Reiter Property

Suppose G admits an invariant mean m on $L^\infty(G)$. Then m restricts to an invariant mean on $UCB(G)$.

Lemma

For all $f \in L^1(G)_{1,+}$ and all $\varphi \in UCB(G)$,

$$m(f * \varphi) = m(\varphi)$$

Assuming the lemma, let $\{f_i\}_i$ be a sequence in $L^1(G)_{1,+}$ such that $\text{supp}(f_i) \rightarrow \{e\}$. Then for all $\varphi \in L^\infty(G)$ and $f \in L^1(G)_{1,+}$

$$\lim_i \|f * f_i * \varphi - f * \varphi\| = 0$$

so since means are continuous and by the result of the lemma

$$m(f * \varphi) = \lim_i m(f * f_i * \varphi) = \lim_i m(f_i * \varphi)$$

Thus for all $\varphi \in L^\infty(G)$ and all $f, f' \in L^1(G)_{1,+}$

$$m(f * \varphi) = m(f' * \varphi) \tag{1}$$

Invariant mean implies Reiter Property

We have $m : UCB(G) \rightarrow \mathbb{R}$ an invariant mean. Fix a function $f_0 \in L^1(G)_{1,+}$. Define a mean $\tilde{m}$ on $L^\infty(G)$ by

$$\tilde{m}(\varphi) = m(f_0 * \varphi)$$

Then $\tilde{m}(\varphi) \geq 0$ if $\varphi \geq 0$ and $\tilde{m}(\chi_G) = m(f_0 * \chi_G) = m(\chi_G) = 1$ by the lemma.

For all $f \in L^1(G)_{1,+}$ and $\varphi \in L^\infty(G)$ we have by (1)

$$\tilde{m}(f * \varphi) = m(f_0 * f * \varphi) = m(f_0 * \varphi) = \tilde{m}(\varphi).$$

That is, $\tilde{m}$ is a **topological invariant mean** on $L^\infty(G)$, meaning that $\tilde{m}$ is a mean so that for all $f \in L^1(G)_{1,+}$ and $\varphi \in L^\infty(G)$

$$\tilde{m}(f * \varphi) = \tilde{m}(\varphi).$$

Invariant mean implies Reiter Property

Since $L^1(G)_{1,+}$ is weak* dense in the set of all means on $L^\infty(G)$, there exists a sequence $\{f_i\}$ in $L^1(G)_{1,+}$ converging to $\tilde{m}$ in the weak* topology. That is, for all $\varphi \in L^\infty(G)$,

$$m_{f_i}(\varphi) \rightarrow \tilde{m}(\varphi)$$

Let $f \in L^1(G)_{1,+}$. From the definition of convolution we get for all $\varphi \in L^\infty(G)$,

$$m_{f*f_i}(\varphi) = m_{f_i}(f * \varphi)$$

thus as $\tilde{m}$ is a topological invariant mean

$$|m_{(f*f_i-f)}(\varphi)| = |m_{f*f_i}(\varphi) - m_{f_i}(\varphi)| \rightarrow |\tilde{m}(f * \varphi) - \tilde{m}(\varphi)| = 0$$

In other words, $(f * f_i - f_i)$ converges to 0 in the weak* topology on $L^1(G)$, where we consider $L^1(G)$ as a subspace of $L^\infty(G)^*$.

Invariant mean implies Reiter Property

Consider

$$E = \prod_{f \in L^1(G)_{1,+}} L^1(G)$$

Since the set $L^1(G)_{1,+}$ is convex, the set E is locally convex.
Define

$$\Sigma = \{(f * g - g)_{f \in L^1(G)_{1,+}} : g \in L^1(G)_{1,+}\} \subset E$$

The set Σ is convex (check), and since $(f * f_i - f_i) \rightarrow 0$ in the weak* topology on each $L^1(G)$, the closure of Σ in the weak* topology on E (which is just the product of the weak* topologies on each $L^1(G)$) contains 0.

Since E is locally convex and Σ is convex, the closure of Σ in the weak* topology is the same as the closure of Σ in the product of the norm topologies. So there exists a sequence $\{g_j\}_j$ in $L^1(G)_{1,+}$ such that for all $f \in L^1(G)_{1,+}$

$$\lim_j \|f * g_j - g_j\|_1 = 0$$

Invariant mean implies Reiter Property

There exists a sequence $\{g_j\}_j$ in $L^1(G)_{1,+}$ such that for all $f \in L^1(G)_{1,+}$

$$\lim_j \|f * g_j - g_j\|_1 = 0 \quad (2)$$

Since each g_j has bounded norm, this holds uniformly for f in compact (w.r.t. norm) subsets of $L^1(G)$.

Let Q be a compact neighbourhood of e . Let $\varepsilon > 0$. Fix $f \in L^1(G)_{1,+}$. Since the map $g \mapsto g \cdot f$ is continuous from G to $L^1(G)$, the set $\{q \cdot f \mid q \in Q\}$ is compact in $L^1(G)$. So by Equation (2) holding uniformly on compact subsets, there exists j such that for all $q \in Q$

$$\|(q \cdot f) * g_j - g_j\|_1 < \varepsilon$$

Put $g = f * g_j$. Note that $q \cdot g = (q \cdot f) * g_j$. Then for all $q \in Q$

$$\|(q \cdot g) - g\|_1 \leq \|(q \cdot f) * g_j - g_j\|_1 + \|(f * g_j) - g_j\|_1 < 2\varepsilon$$

We have established the Reiter Property.