

# The Banach–Tarski Paradox and Amenability

## Lecture 21: Fixed Point Properties and Amenability

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# Invariant means, Reiter's Property, the Følner Condition

Let  $G$  be a locally compact group. We have proved that the following are equivalent, and so can all be taken as the definition of amenability.

- ▶  $G$  admits an **invariant mean**, i.e., a linear functional  $m : L^\infty(G) \rightarrow \mathbb{R}$  such that:
  1.  $m(f) \geq 0$  if  $f \geq 0$
  2.  $m(\chi_G) = 1$
  3.  $m(g \cdot f) = m(f)$  for all  $g \in G$  and  $f \in L^\infty(G)$
- ▶  $G$  satisfies **Reiter's Property**, i.e.,  $\forall \varepsilon > 0$  and  $\forall$  compact  $K \subset G, \exists f \in L^1(G)_{1,+}$  s. t.  $\forall x \in K,$

$$\|x \cdot f - f\|_1 \leq \varepsilon$$

- ▶  $G$  satisfies the **Følner Condition**, i.e.,  $\forall \varepsilon > 0$  and  $\forall$  compact  $K \subset G, \exists$  a Borel set  $U \subseteq G$  with  $0 < \mu(U) < \infty$  s.t.  $\forall x \in K$

$$\mu(xU \Delta U) < \varepsilon \mu(U)$$

# Convexity and local convexity

## Definitions

1. A **topological vector space**  $V$  is a vector space which is also a Hausdorff topological space such that vector addition and scalar multiplication are continuous. For example, Banach spaces are topological vector spaces.
2. A subset  $X$  of a topological vector space  $V$  is **convex** if for all  $x, y \in X$  and  $t \in [0, 1]$ , the point  $tx + (1 - t)y$  is in  $X$ .
3. A subset  $U$  of a topological vector space  $V$  over  $k = \mathbb{R}$  or  $\mathbb{C}$  is **balanced** if for all  $x \in U$  and  $c \in k$  with  $|c| \leq 1$ , the point  $cx$  is in  $U$ .
4. A topological vector space  $V$  is **locally convex** if every neighbourhood of  $0$  contains a convex balanced neighbourhood of  $0$ .

If  $k = \mathbb{R}$  then  $V$  is locally convex if and only if every neighbourhood of  $0$  contains a convex neighbourhood of  $0$ .

## Fixed point characterisation of amenable groups

Let  $X$  be a convex subset of a topological vector space  $V$ . A mapping  $\alpha : X \rightarrow X$  is **affine** if for all  $x, y \in X$  and  $0 \leq t \leq 1$

$$\alpha(tx + (1 - t)y) = t\alpha(x) + (1 - t)\alpha(y)$$

Let  $G$  be a locally compact group. A continuous action of  $G$  on  $X$  is an **affine action** if  $x \mapsto gx$  is an affine mapping for all  $g \in G$ .

### Example

Any isometry  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  is affine. The action of  $O(n, \mathbb{R})$  on  $\mathbb{R}^n$  is affine.

Today we will prove:

### Theorem (Day)

*A locally compact group  $G$  is amenable if and only if any continuous affine action of  $G$  on a nonempty compact convex subset of a locally convex topological vector space has a fixed point.*

## Fixed point property implies amenable

Suppose that any continuous affine action of  $G$  on a nonempty compact convex subset of a locally convex topological vector space has a fixed point.

### Definition

Let  $UCB(G)$  be the subspace of  $L^\infty(G)$  given by

$$UCB(G) = \{f \in L^\infty(G) \mid g \mapsto g \cdot f \text{ is continuous}\}$$

The  $G$ -action on  $L^\infty(G)$  preserves  $UCB(G)$ . If  $G$  is compact or discrete, then  $L^\infty(G) = UCB(G)$ .

Let  $UCB(G)^*$  be the dual of  $UCB(G)$ , with the weak\*-topology. Then  $UCB(G)^*$  is a locally convex topological vector space.

# Means on $UCB(G)$

## Definition

A **mean** on  $UCB(G)$  is a linear functional  $m : UCB(G) \rightarrow \mathbb{R}$  s.t.:

1.  $m(\varphi) \geq 0$  if  $\varphi \geq 0$
2.  $m(\chi_G) = 1$

For example, the evaluation map  $\varphi \mapsto \varphi(e)$  is a mean on  $UCB(G)$ .

We proved the following in Lecture 19 for means on  $L^\infty(G)$ . The same proof works for  $UCB(G)$ .

## Lemma

*Let  $\mathcal{M}$  be the set of all means on  $UCB(G)$ . Then  $\mathcal{M}$  is a subset of the unit ball in  $UCB(G)^*$ .*

We also have:

## Lemma

*$\mathcal{M}$  is closed in the weak\* topology.*

It follows that  $\mathcal{M}$  is a compact convex subset of the unit ball in  $UCB(G)^*$ .

## Action of $G$ on set of means on $UCB(G)^*$

There is a continuous affine action of  $G$  on  $\mathcal{M}$  given by

$$gm(\varphi) = m(g \cdot \varphi)$$

for all  $g \in G$ ,  $m \in \mathcal{M}$  and  $\varphi \in UCB(G)$ . By assumption, this action has a fixed point  $m_0 \in \mathcal{M}$ . That is, for all  $g \in G$  and  $\varphi \in UCB(G)$ ,

$$m_0(g \cdot \varphi) = m_0(\varphi).$$

Hence  $m_0$  is an **invariant** mean on  $UCB(G)$ .

In Lecture 20, we showed that an invariant mean on  $UCB(G)$  implies Reiter's Property. Combined with Lectures 18 and 19 (or more directly), we obtain an invariant mean on  $L^\infty(G)$ . Therefore  $G$  is amenable.

# Amenable implies fixed point property

As motivation:

## Lemma

*Any action of a finite group  $G$  by isometries on  $\mathbb{R}^n$  has a fixed point.*

## Proof.

The fixed point is the centre of mass of a  $G$ -orbit. □

Assume that  $G$  is amenable, and that there is a continuous affine action of  $G$  on a nonempty compact convex subset  $X$  of a topological vector space  $V$ .

We associate to any mean  $m$  on  $UCB(G)$  a **barycentre**  $b_m \in X$  such that  $b_{gm} = gb_m$  for every  $g \in G$ . Then if  $m$  is an invariant mean,  $b_m$  is a fixed point for the action of  $G$  on  $X$ .

To define  $b_m$ , fix a point  $x_0$  in  $X$  and let  $t : G \rightarrow X$  be given by  $t(g) = gx_0$ . The barycentre  $b_m$  is the unique element of  $X$  such that  $\varphi(b_m) = m(\varphi \circ t)$  for every  $\varphi \in V^*$ . If  $m = \delta_g$  the evaluation map, then  $b_m = gx_0$ .

# The Markov–Kakutani fixed point theorem

## Theorem (Markov–Kakutani)

*Let  $G$  be an abelian locally compact group. Then any continuous affine action of  $G$  on a nonempty compact convex subset  $X$  of a locally convex topological vector space  $V$  has a fixed point.*

This provides an alternative proof of the amenability of abelian groups.

To prove this, define for  $n \geq 0$  and  $g \in G$  the map  $A_n(g) : X \rightarrow X$  by

$$A_n(g)(x) = \frac{1}{n+1} \sum_{i=0}^n g^i x$$

Then  $A_n(g)$  is a continuous affine transformation of  $X$ . Let  $\mathcal{A}$  be the semigroup of continuous affine transformations of  $X$  generated by the maps  $A_n(g)$ . Since  $X$  is compact,  $\alpha(X)$  is a closed subset of  $X$  for each  $\alpha \in \mathcal{A}$ .

# The Markov–Kakutani fixed point theorem

Define for  $n \geq 0$  and  $g \in G$

$$A_n(g)(x) = \frac{1}{n+1} \sum_{i=0}^n g^i x$$

Let  $\mathcal{A}$  be the semigroup generated by the maps  $A_n(g)$ . We claim that

$$\bigcap_{\alpha \in \mathcal{A}} \alpha(X)$$

is nonempty. Since  $X$  is compact,  $X$  is covered by finitely many sets of the form  $\alpha(X)$  so it is enough to show that

$\alpha_1(X) \cap \cdots \cap \alpha_n(X)$  is nonempty for all  $\alpha_1, \dots, \alpha_n \in \mathcal{A}$ . Let  $\alpha = \alpha_1 \alpha_1 \cdots \alpha_n \in \mathcal{A}$ . Since  $\mathcal{A}$  is abelian,  $\alpha(X) \subset \alpha_i(X)$  for each  $1 \leq i \leq n$ . So  $\alpha_1(X) \cap \cdots \cap \alpha_n(X)$  contains  $\alpha(X)$  and is thus nonempty.

## The Markov–Kakutani fixed point theorem

Define for  $n \geq 0$  and  $g \in G$

$$A_n(g)(x) = \frac{1}{n+1} \sum_{i=0}^n g^i x$$

Let  $\mathcal{A}$  be the semigroup generated by the maps  $A_n(g)$ . Let

$$x_0 \in \bigcap_{\alpha \in \mathcal{A}} \alpha(X)$$

We claim that  $x_0$  is a fixed point for  $G$ . By our choice of  $x_0$ , for every  $n \geq 0$  and  $g \in G$  there is an  $x \in X$  such that  $x_0 = A_n(g)(x)$ .

Thus for every  $\varphi \in V^*$  and  $g \in G$ ,

$$\begin{aligned} |\varphi(x_0 - gx_0)| &= |\varphi(A_n(g)(x) - gA_n(g)(x))| \\ &= \frac{1}{n+1} |\varphi(x) - \varphi(g^{n+1}x)| \\ &\leq \frac{2C}{n+1} \end{aligned}$$

where  $C = \sup_{y \in X} |\varphi(y)| < \infty$  since  $X$  is compact.

# The Markov–Kakutani fixed point theorem

We have shown: for every  $\varphi \in V^*$ ,  $n \geq 0$  and  $g \in G$ ,

$$|\varphi(x_0 - gx_0)| \leq \frac{2C}{n+1}$$

where  $C = \sup_{y \in X} |\varphi(y)| < \infty$  since  $X$  is compact. Thus  $\varphi(x_0) = \varphi(gx_0)$  for every  $\varphi \in V^*$ . Hence  $x_0 = gx_0$  for all  $g \in G$ , as required. That is,  $G$  has a fixed point in  $X$ .

## Amenability of group extensions

As a corollary of the fixed point characterisation of amenability, we get:

### Corollary

*Let  $G$  be a locally compact group and let  $N$  be a closed normal subgroup of  $G$ . If  $N$  and  $G/N$  are amenable, then  $G$  is amenable.*

### Proof.

Assume that a continuous affine action of  $G$  on a nonempty compact convex set  $X$  is given. Let  $X^N$  be the set of all fixed points of  $N$  in  $X$ . The set  $X^N$  is nonempty since  $N$  is amenable. Also  $X^N$  is closed, compact and convex. The set  $X^N$  is preserved by the action of  $G$  since if  $x \in X^N$  then for all  $g \in G$ ,  $n \in N$

$$ngx = gx \iff g^{-1}ngx = x$$

and  $g^{-1}ng \in N$ . The action of  $G$  on  $X^N$  then factors to an action of  $G/N$  on  $X^N$ , which has a fixed point  $x_0$  since  $G/N$  is amenable. But then  $x_0$  is fixed by each  $g \in G$ . So  $G$  is amenable.

## Examples: isometries of $\mathbb{R}^n$ and orthogonal groups

1. Let  $G = \text{Isom}(\mathbb{R}^n)$ . Then  $G$  has a closed normal subgroup  $N = \mathbb{R}^n$ , the group of translations. Since  $N$  is abelian,  $N$  is amenable. You essentially showed in Assignment 1 that  $G/N \cong O(n, \mathbb{R})$ . Since  $O(n, \mathbb{R})$  is compact,  $G/N$  is amenable. Therefore  $\text{Isom}(\mathbb{R}^n)$  is amenable as a **non-discrete** group for all  $n$ .
2. Let  $G = O(2, \mathbb{R})$ . Then  $G$  has a closed normal subgroup  $N = SO(2, \mathbb{R}) \cong \mathbb{T}$ . Since  $N$  is abelian,  $N$  is amenable. Now  $N$  has index 2 in  $G$  so  $G/N$  is the finite cyclic group  $C_2$ . Since  $C_2$  is finite,  $C_2$  is amenable. Therefore  $O(2, \mathbb{R})$  is amenable as a **discrete** group. It follows that  $\text{Isom}(\mathbb{R}^2)$  is amenable as a discrete group. An even easier argument shows that  $\text{Isom}(\mathbb{R})$  is amenable as a discrete group.
3. For  $n \geq 3$ , the groups  $O(n, \mathbb{R})$  and thus  $\text{Isom}(\mathbb{R}^n)$  contain an  $F_2$  subgroup, and so admit a paradoxical decomposition. Therefore for  $n \geq 3$ , the groups  $O(n, \mathbb{R})$  and  $\text{Isom}(\mathbb{R}^n)$  are NOT amenable as discrete groups.