# The Banach–Tarski Paradox and Amenability
# Lecture 22: Solvable Groups and Amenability

18 October 2011

# Fixed point characterisation of amenable groups

In Lecture 21 we proved/sketched:

### Theorem (Day)

*A locally compact group $G$ is amenable if and only if any continuous affine action of $G$ on a nonempty compact convex subset of a locally convex topological vector space has a fixed point.*

and using this we proved:

### Corollary

*Let $G$ be a locally compact group and let $N$ be a closed normal subgroup of $G$. If $N$ and $G/N$ are amenable, then $G$ is amenable.*

Today we will use this corollary to show that solvable groups are amenable. The terminology comes Galois theory. Specifically, $S_n$, the symmetric group on $n$ letters, is solvable if and only if $n \leq 4$.

Solvable (US) = soluble (UK), take your pick.

# Nilpotent groups

Let $G$ be a group. Define $G_1 = G$ and for all $n \geq 1$ define

$$G_{n+1} = [G_n, G]$$

Then $G_2 = [G, G] = G'$ and

$$G = G_1 \geq G_2 \geq G_3 \geq \cdots$$

## Definition

A group $G$ is nilpotent if there is an integer $n$ such that $G_{n+1} = \{1\}$. That is, $G$ is nilpotent if and only if the series

$$G \geq [G, G] \geq [[G, G], G] \geq \cdots$$

terminates at the trivial group after finitely many steps.

# Nilpotent groups

## Examples

1. If $G$ is abelian then $G$ is nilpotent, since $[G, G] = 1$.
2. The discrete Heisenberg group is nilpotent. So is the continuous version (replace $a, b, c \in \mathbb{Z}$ with $x, y, z \in \mathbb{R}$).
3. The symmetric group $G = S_3$ is not nilpotent. We compute

$$G_2 = [G, G] = \langle (123) \rangle$$

but

$$G_3 = [G_2, G] = G_2$$

and so for all $n \geq 3$, $G_n = G_2 \neq \{1\}$.

# Nilpotent groups and polynomial growth

**Theorem (Dixmier, Milnor, Wolf 1960s)**

*Finitely generated nilpotent groups have polynomial growth.*

**Corollary**

*Every nilpotent group $G$ is amenable (as a discrete group).*

**Theorem (Gromov 1980)**

*If a finitely generated group $G$ has polynomial growth then $G$ has a finite index nilpotent subgroup.*

# Solvable groups

Let $G$ be a group. A subnormal series is a sequence of subgroups

$$G = H_1 \triangleright H_2 \triangleright \cdots \triangleright H_n = 1$$

The groups $H_i$ are not required to be normal in $G$. The quotients $H_i/H_{i+1}$ are called the factor groups.

### Example

$S_n$ has subnormal series

$$S_n \triangleright A_n \triangleright 1$$

with factor groups $S_n/A_n \cong C_2$ and $A_n/1 \cong A_n$.

### Definition

A group $G$ is solvable if it has a subnormal series whose factor groups are all abelian.

# Solvable groups

Let $G$ be a group. Define $G^{(1)} = G$ and for all $n \geq 2$ define

$$G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$$

Then $G^{(2)} = [G, G] = G'$ and the series

$$G = G^{(1)} \triangleright G^{(2)} \triangleright G^{(3)} \triangleright \cdots$$

has all factor groups abelian. If $G^{(n)} = 1$ for some $n$, $G$ is solvable. Conversely, suppose $G$ is solvable with

$$G = H_1 \triangleright H_2 \triangleright \cdots \triangleright H_n = 1$$

having all factor groups abelian. By induction on $i$, $H_i$ contains $G^{(i)}$. So $G^{(n)} = 1$.

## Definition

A group $G$ is solvable if there is an integer $n$ such that $G^{(n)} = \{1\}$. That is, $G$ is solvable if and only if the series

$$G \triangleright [G, G] \triangleright [[G, G], [G, G]] \triangleright \cdots$$

terminates at the trivial group after finitely many steps.

# Relationship to nilpotence

### Definition

A group $G$ is nilpotent if and only if the series

$$G \geq [G, G] \geq [[G, G], G] \geq \cdots$$

terminates at the trivial group after finitely many steps.

### Definition

A group $G$ is solvable if and only if the series

$$G \triangleright [G, G] \triangleright [[G, G], [G, G]] \triangleright \cdots$$

terminates at the trivial group after finitely many steps.

### Lemma

*If $G$ is nilpotent then $G$ is solvable.*

# Examples

1. You can check by direct calculation that $S_2$, $S_3$ and $S_4$ are solvable, and that $S_3$ and $S_4$ are not nilpotent. The essential reason $S_n$ is not solvable for $n \geq 5$ is that $A_n$ is simple for $n \geq 5$.
2. A subnormal series for $G = O(2, \mathbb{R})$ with all factor groups abelian is
$$G = O(2, \mathbb{R}) \triangleright SO(2, \mathbb{R}) \triangleright 1$$
thus $O(2, \mathbb{R})$ is a solvable group.
3. On Assignment 3, you will show that
$$G = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} : x, y \in \mathbb{R}, x \neq 0 \right\}$$
and
$$G = BS(1, n) = \langle a, b \mid aba^{-1} = b^n \rangle$$
are solvable but not nilpotent.

# Solvability of subgroups and quotients

**Lemma**

*If $G$ is solvable then every subgroup $H$ of $G$ is solvable and every quotient $Q$ of $G$ is solvable.*

**Proof.**

By induction on $n$, $H^{(n)} \leq G^{(n)}$.
Let $f : G \to Q$ be a surjective group homomorphism. Suppose

$$G = H_1 \triangleright H_2 \triangleright \cdots \triangleright H_n = 1$$

is a subnormal series with all factor groups abelian. It is routine to check that

$$Q = f(H_1) \triangleright f(H_2) \triangleright \cdots \triangleright f(H_n) = 1$$

is a subnormal series with all factor groups abelian. □

# Solvable groups are amenable

### Corollary

*Solvable groups are amenable (as discrete groups).*

### Proof.

Suppose

$$G = G^{(1)} \triangleright G^{(2)} \triangleright G^{(3)} \triangleright \cdots \triangleright G^{(n)} = 1$$

Carry out induction on $n \geq 2$. If $n = 2$ then $G$ is abelian hence amenable. If $n > 2$ then $G^{(1)}/G^{(2)} = G/[G, G]$ is abelian hence amenable and $G^{(2)}$ is amenable by inductive assumption, so $G$ is amenable. □

# Solvability of group extensions

## Lemma

*If $N \triangleleft G$ and if $N$ and $G/N$ are solvable, then $G$ is solvable.*

## Proof.

Let

$$G/N = H_1^* \triangleright H_2^* \triangleright \cdots \triangleright H_n^* = 1$$

be a subnormal series with all factor groups abelian. By isomorphism theorems, $H_i^* = H_i/N$ where $N \triangleleft H_i \leq G$, and $H_i/H_{i+1} \cong H_i^*/H_{i+1}^*$ is abelian. So a subnormal series for $G$ with all factor groups abelian is obtained by starting with

$$G = H_1 \triangleright H_2 \triangleright \cdots \triangleright H_n = N$$

and then continuing down from $N$. □

## Corollary

$\mathrm{Isom}(\mathbb{R}^n)$ *is solvable for* $n = 1, 2$. *Hence* $\mathrm{Isom}(\mathbb{R}^n)$ *is amenable for* $n = 1, 2$.

# Solvability and growth

The following theorems are used in the proof of the Gromov Polynomial Growth Theorem.

### Theorem (Milnor–Wolf)

*A finitely generated solvable group $G$ has exponential growth unless $G$ contains a nilpotent subgroup of finite index.*

Thus since nilpotent groups have polynomial growth, a finitely generated solvable group $G$ has either exponential or polynomial growth, and the latter occurs if and only if $G$ has a finite index nilpotent subgroup.

# The Tits Alternative

## Theorem (Tits Alternative)

*Let $L$ be a Lie group with finitely many connected components. Let $G$ be a finitely generated subgroup of $L$. Then exactly one of the following occurs:*

1. *$G$ contains a free group of rank 2.*
2. *$G$ contains a solvable subgroup of finite index.*

Note that:

1. If $G$ contains a free group of rank 2, then $G$ has exponential growth.
2. If $G$ contains a solvable subgroup of finite index, then by the Milnor–Wolf Theorem, $G$ has exponential growth unless $G$ has a finite index nilpotent subgroup.
3. If $G$ contains a solvable subgroup of finite index, then $G$ is amenable.

The proof of the Tits Alternative uses the Ping-Pong Lemma from Assignment 1.