

The Banach–Tarski Paradox and Amenability

Lecture 23: Unitary Representations and Amenability

23 October 2012

Subgroups of amenable groups are amenable

One of today's aims is to prove:

Theorem

Let G be a locally compact group and let H be a closed subgroup of G . If G is amenable then H is amenable.

Corollary

Let G be a discrete group. If G is amenable then every subgroup of G is amenable.

To prove this, we will establish yet another characterisation of amenability, using unitary representations. We will also (briefly) discuss the relationship between amenability and Kazhdan's Property (T).

Obstruction to amenability

Corollary

Let G be a discrete group. If G contains a free group of rank 2, then G is not amenable (as a discrete group).

For example, the following groups are *not* amenable (as discrete groups):

- ▶ $SO(3, \mathbb{R})$ and thus $O(n, \mathbb{R})$ and $SO(n, \mathbb{R})$, for $n \geq 3$
- ▶ $SL(2, \mathbb{Z})$ and thus $SL(n, \mathbb{Z})$ for $n \geq 2$, $SL(n, \mathbb{R})$ for $n \geq 2$, $GL(n, \mathbb{R})$ for $n \geq 2$, etc
- ▶ Any finitely generated subgroup G of a Lie group with finitely many connected components such that G is not virtually solvable, by the Tits Alternative.

Unitary representations

In this lecture, Hilbert spaces are always over $\mathbb{C}$. Denote by

$$\langle \xi, \eta \rangle$$

the inner product in such a space $\mathcal{H}$.

The **unitary group** $\mathcal{U}(\mathcal{H})$ of $\mathcal{H}$ is the group of all invertible bounded linear operators $U : \mathcal{H} \rightarrow \mathcal{H}$ which are **unitary**, meaning that for all $\xi, \eta \in \mathcal{H}$

$$\langle U\xi, U\eta \rangle = \langle \xi, \eta \rangle$$

or equivalently $U^*U = UU^* = I$ where U^* is the adjoint.

Let G be a locally compact group. A **unitary representation** of G in $\mathcal{H}$ is a group homomorphism $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ which is **strongly continuous**, that is, $g \mapsto \pi(g)\xi$ is continuous from G to $\mathcal{H}$ for each $\xi \in \mathcal{H}$. Write $(\pi, \mathcal{H})$ for such a representation.

Examples of unitary representations

1. Let $\mathcal{H} = \{*\}$ the one-point space and denote by 1_G the **unit representation** of G . That is, for all $g \in G$

$$1_G(g)^* = *$$

This representation is clearly unitary.

2. Let $\mathcal{H} = L^2(G)$. Denote by λ_G the **left-regular representation**

$$(\lambda_G(g)f)(x) = (g \cdot f)(x) = f(g^{-1}x)$$

for all $g \in G$, $f \in L^2(G)$, $x \in G$. Then for all $g \in G$, $f_1, f_2 \in \mathcal{H}$, since Haar measure μ is G -invariant

$$\langle \lambda_G(g)f_1, \lambda_G(g)f_2 \rangle = \int_G (g \cdot f_1) \overline{(g \cdot f_2)} d\mu = \int_G f_1 \overline{f_2} d\mu = \langle f_1, f_2 \rangle$$

so λ_G is unitary. Also the map $G \rightarrow L^2(G)$ given by $g \mapsto (g \cdot f)$ is continuous for each $f \in L^2(G)$, so λ_G is a unitary representation.

Invariant vectors

Definition

The unitary representation $(\pi, \mathcal{H})$ has non-zero **invariant vectors** if there exists $\xi \neq 0$ in $\mathcal{H}$ such that $\pi(g)\xi = \xi$ for all $g \in G$. If this holds, we write $1_G \subset \pi$.

Example

Let G be a compact group. Then $(\lambda_G, L^2(G))$ has non-zero invariant vector χ_G .

Almost invariant vectors

Definition

Let $(\pi, \mathcal{H})$ be a unitary representation. For a subset $Q \subset G$ and $\varepsilon > 0$, a vector $\xi \in \mathcal{H}$ is **(Q, ε) -invariant** if

$$\sup_{x \in Q} \|\pi(x)\xi - \xi\| < \varepsilon \|\xi\|$$

The representation $(\pi, \mathcal{H})$ **almost has invariant vectors** if it has non-zero (Q, ε) -invariant vectors for every *compact* $Q \subset G$ and every $\varepsilon > 0$. If this holds, we write $1_G \preceq \pi$.

Example

If $(\pi, \mathcal{H})$ has a non-zero invariant vector ξ then ξ is (Q, ε) -invariant for every $Q \subset G$ and $\varepsilon > 0$, so π almost has invariant vectors.

Almost invariant vectors for $G = \mathbb{R}$

Let $G = (\mathbb{R}, +)$ with its usual topology and let $\mathcal{H} = L^2(\mathbb{R})$. Then the regular representation $\lambda_{\mathbb{R}}$ on $\mathcal{H}$ almost has invariant vectors. Let Q be a compact subset of $\mathbb{R}$ and let $\varepsilon > 0$. Choose $a < b$ such that for all $x \in Q$

$$\frac{2|x|}{b-a} < \varepsilon^2$$

Let

$$\xi = \frac{1}{\sqrt{b-a}} \chi_{[a,b]}$$

Then ξ is a unit vector in $\mathcal{H}$, and for all $x \in Q$

$$\|\lambda_{\mathbb{R}}(x)\xi - \xi\|_2^2 = \frac{1}{b-a} \int_{\mathbb{R}} (\chi_{[a+x,b+x]} - \chi_{[a,b]})^2 d\mu = \frac{2|x|}{b-a} < \varepsilon^2$$

thus

$$\sup_{x \in Q} \|\lambda_G(x)\xi - \xi\| < \varepsilon \|\xi\|$$

as required.

Amenability and almost invariant vectors

Theorem (Hulanicki–Reiter)

A locally compact group G is amenable if and only if $1_G \preceq \lambda_G$.

That is, G is amenable if and only if the left-regular representation $(\lambda_G, L^2(G))$ almost has invariant vectors, meaning that for every compact $Q \subset G$ and every $\varepsilon > 0$, there is a $0 \neq \xi \in L^2(G)$ such that

$$\sup_{x \in Q} \|\lambda_G(x)\xi - \xi\| < \varepsilon \|\xi\|$$

Examples

Compact groups (thus finite groups), and the group $(\mathbb{R}, +)$ with its usual topology, are amenable.

We will show that $1_G \preceq \lambda_G$ is equivalent to Reiter's Property:

Definition

A locally compact group G satisfies **Reiter's Property** if for every $\varepsilon > 0$ and every compact $Q \subset G$, there is an $f \in L^1(G)_{1,+}$ such that $\sup_{x \in Q} \|x \cdot f - f\|_1 \leq \varepsilon$.

Amenability and almost invariant vectors

Theorem (Hulanicki–Reiter)

A locally compact group G is amenable if and only if $1_G \preceq \lambda_G$.

Suppose that $1_G \preceq \lambda_G$. Then given compact $Q \subset G$ and $\varepsilon > 0$, there exists $f = f_{Q,\varepsilon} \in L^2(G)$ such that $\|f\|_2 = 1$ and

$$\sup_{x \in Q} \|\lambda_G(x)f - f\|_2 < \varepsilon$$

Put $g = |f|^2 = \bar{f}f$. Then $g \in L^1(G)_{1,+}$ and by the Cauchy–Schwarz inequality, for all $x \in Q$,

$$\begin{aligned} \|x \cdot g - g\|_1 &\leq \|\lambda_G(x)\bar{f} + \bar{f}\|_2 \|\lambda_G(x)f - f\|_2 \\ &\leq 2\|\lambda_G(x)f - f\|_2 \\ &< \varepsilon \end{aligned}$$

so the Reiter Property holds.

Amenability and almost invariant vectors

Conversely suppose that the Reiter Property holds. Then given compact $Q \subset G$ and $\varepsilon > 0$, let $f \in L^1(G)_{1,+}$ be such that

$$\sup_{x \in Q} \|x \cdot f - f\|_1 \leq \varepsilon$$

Let $g = \sqrt{f}$. Then $g \in L^2(G)$ and $\|g\|_2 = 1$. Now for all $x \in Q$

$$\begin{aligned} \|\lambda_G(x)g - g\|_2^2 &= \int_G |g(x^{-1}y) - g(y)|^2 d\mu(y) \\ &\leq \int_G |(g(x^{-1}y))^2 - (g(y))^2| d\mu(y) \\ &= \|x \cdot f - f\|_1 \\ &< \varepsilon \end{aligned}$$

so $1_G \preceq \lambda_G$ as required.

Subgroups of amenable groups

We will now use

Theorem (Hulanicki–Reiter)

A locally compact group G is amenable if and only if $1_G \preceq \lambda_G$.

to prove:

Theorem

Let G be a locally compact group and H a closed subgroup of G . If G is amenable then H is amenable.

Subgroups of amenable groups

Definition

Let $(\pi, \mathcal{H})$ and $(\rho, \mathcal{K})$ be unitary representations of a locally compact group G . We say that π is **weakly contained** in ρ , denoted $\pi \preceq \rho$, if for every $\xi \in \mathcal{H}$, every compact subset $Q \subset G$ and every $\varepsilon > 0$, there are $\eta_1, \dots, \eta_n \in \mathcal{K}$ such that for all $x \in Q$

$$\left| \langle \pi(x)\xi, \xi \rangle - \sum_{i=1}^n \langle \rho(x)\eta_i, \eta_i \rangle \right| < \varepsilon$$

We will use but not prove:

Proposition

1. If π , ρ and σ are unitary representations of G , such that $\pi \preceq \rho$ and $\rho \preceq \sigma$, then $\pi \preceq \sigma$.
2. 1_G is weakly contained in π if and only if π almost has invariant vectors (thus our earlier notation $1_G \preceq \pi$ is justified).

Subgroups of amenable groups

Proposition

If H is a closed subgroup of a locally compact group G , then $\lambda_G|_H \preceq \lambda_H$, where $\lambda_G|_H$ is the restriction of λ_G to H .

Proof.

We just sketch the case G discrete, where H is any subgroup. Let T be a set of representatives of the right cosets $H \backslash G$. Then $L^2(G)$ has a direct sum decomposition

$$L^2(G) = \oplus_{t \in T} L^2(Ht)$$

where each $L^2(Ht)$ is a $\lambda_G|_H$ -invariant subspace. Now the restriction of $\lambda_G|_H$ to each $L^2(Ht)$ is equivalent to the representation $(\lambda_H, L^2(H))$. This implies the weak containment. □

Subgroups of amenable groups

Theorem

Let G be a locally compact group and H a closed subgroup of G . If G is amenable then H is amenable.

Proof.

We want to show that $1_H \preceq \lambda_H$, given that H is a closed subgroup of a locally compact group G such that $1_G \preceq \lambda_G$. By the propositions above, it suffices to prove that $1_H \preceq \lambda_G|_H$. But this follows from the fact that a compact subset of H is compact in G , and the definition of weak containment applied to $\pi = 1_H$ or 1_G and $\rho = \lambda_G|_H$ or λ_G , respectively. □

Kazhdan's Property (T)

Definition (Kazhdan 1967)

A locally compact group G has **Property (T)** if there is a compact subset $Q \subset G$ and an $\varepsilon > 0$ such that, whenever a unitary representation $(\pi, \mathcal{H})$ has a (Q, ε) -invariant vector, then π has a non-zero invariant vector.

Example

Compact groups have Property (T).

As with amenability, there are *many* other formulations of Kazhdan's Property (T). The original motivation for Kazhdan was to answer (in the affirmative) the following question of Siegel from the late 1940s:

Question

Is $SL(n, \mathbb{Z})$ finitely generated for $n \geq 3$?

Kazhdan's Property (T) and amenability

Property (T) can be thought of as a strong negation of amenability.

Proposition

Let G be a locally compact group which has Property (T) and is amenable. Then G is compact. In particular, a discrete group which has Property (T) and is amenable must be finite.

Proof.

Since G is amenable, λ_G almost has invariant vectors. Since G has Property (T), λ_G has a non-zero invariant vector. Thus G is compact. □

This proposition is the strategy used to prove the following special case of Margulis' Normal Subgroup Theorem (1970s).

Theorem

Let N be a normal subgroup of $\Gamma = SL(n, \mathbb{Z})$, $n \geq 3$. If N is infinite then Γ/N is finite.