

The Banach–Tarski Paradox and Amenability

Lecture 24: Elementary Amenable Groups and the von Neumann Conjecture

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Elementary amenable groups

For G a group, we have shown that:

- ▶ if G is amenable then every quotient of G by a closed normal subgroup is amenable
- ▶ if G is amenable then every closed subgroup of G is amenable
- ▶ if N is a closed normal subgroup of G , and N and G/N are amenable, then G is amenable
- ▶ if G is finite or abelian then G is amenable

We will soon define a class of groups called **elementary amenable groups**, which are roughly speaking those which can be obtained by starting with finite and abelian groups and then applying some finite sequence of the above operations (as well as another operation defined today).

Short exact sequences

A **short exact sequence** of groups A , B and C is a sequence of groups and group homomorphisms

$$1 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \longrightarrow 1$$

where

- ▶ the map $1 \rightarrow A$ is inclusion of the trivial group
- ▶ ψ is injective
- ▶ $\ker \phi = \text{image } \psi$
- ▶ ϕ is surjective
- ▶ the map $C \rightarrow 1$ is the trivial homomorphism

By the first isomorphism theorem, $B/\ker(\phi) \cong C$. Since ψ is injective we can identify A with its image $\psi(A)$ in B . Since $\psi(A) = \ker(\phi)$, we have that A is a normal subgroup of B . So we have $B/A \cong C$.

Group extensions

From now on we may write short exact sequences of groups as

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

with $N \triangleleft G$ and $G/N \cong Q$.

Definition

If there is a short exact sequence as above, then the group G is an **extension** of N by Q .

So we have:

Theorem

The class of amenable groups is closed under group extensions (so long as N is closed in G).

Examples of group extensions

1. For any groups A and B , we can form short exact sequences

$$1 \longrightarrow A \longrightarrow A \times B \longrightarrow B \longrightarrow 1$$

and

$$1 \longrightarrow B \longrightarrow A \times B \longrightarrow A \longrightarrow 1$$

So the direct product $A \times B$ is an extension of B by A and an extension of A by B . Thus direct products of finitely many amenable groups are amenable.

2. Examples of group extensions which are not direct products are

$$1 \longrightarrow C_n \longrightarrow D_{2n} \longrightarrow C_2 \longrightarrow 1$$

where C_n is cyclic of order n and D_{2n} is dihedral of order $2n$ and

$$1 \longrightarrow SO(2, \mathbb{R}) \longrightarrow O(2, \mathbb{R}) \longrightarrow C_2 \longrightarrow 1$$

Semidirect products

Let G be a group with subgroups N and H such that

- ▶ $N \triangleleft G$
- ▶ $G = NH$, that is, every element $g \in G$ can be expressed as $g = nh$ where $n \in N$ and $h \in H$
- ▶ $N \cap H = 1$, hence every $g \in G$ can be expressed *uniquely* as $g = nh$ where $n \in N$ and $h \in H$

Then there is a short exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1$$

and G is called the (internal) **semidirect product** of N by H , denoted $G = N \rtimes H$.

Example

$G = \text{Isom}(\mathbb{R}^n)$, $N = \{\text{translations}\} \cong \mathbb{R}^n$ and $H = O(n, \mathbb{R})$. Thus as a non-discrete group $\text{Isom}(\mathbb{R}^n)$ is amenable for all n , and as a discrete group $\text{Isom}(\mathbb{R}^n)$ is amenable for $n = 1, 2$.

Semidirect products

Let G be a group with subgroups N and H such that

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- ▶ $G = NH$
- ▶ $N \cap H = 1$.

Then there is a short exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1$$

and G is called the (internal) **semidirect product** of N by H , denoted $G = N \rtimes H$.

Note that if $g_1 = n_1 h_1$ and $g_2 = n_2 h_2$ where $n_1, n_2 \in N$ and $h_1, h_2 \in H$

$$g_1 g_2 = n_1 h_1 n_2 h_2 = n_1 h_1 n_2 h_1^{-1} h_1 h_2 = n_1 n'_2 h_1 h_2$$

where $n'_2 = h_1 n_2 h_1^{-1} \in N$. Since H normalises N , there is a homomorphism $\varphi : H \rightarrow \text{Aut}(N)$ given by $\varphi(h)(n) = hnh^{-1}$. So we may write multiplication in G by

$$n_1 h_1 n_2 h_2 = n_1 \varphi(h_1)(n_2) h_1 h_2$$

Semidirect products

Suppose we are given groups N and H together with a homomorphism $\varphi : H \rightarrow \text{Aut}(N)$. That is, H acts on N . Then we may form the (external) **semidirect product**, denoted $G = N \rtimes_{\varphi} H$.

The elements of $G = N \rtimes_{\varphi} H$ are pairs (n, h) where $n \in N$ and $h \in H$. Multiplication in the group is given by

$$(n_1, h_1)(n_2, h_2) = (n_1\varphi(h_1)(n_2), h_1h_2)$$

You should check that the set $\{(n, h) : n \in N, h \in H\}$ satisfies the axioms for a group.

The group G has a normal subgroup isomorphic to N , given by $\{(n, 1_H) : n \in N\}$, and a subgroup (not necessarily normal) isomorphic to H , given by $\{(1_N, h) : h \in H\}$. There is a short exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1$$

Examples of semidirect products

1. $D_{2n} = C_n \rtimes_{\varphi} C_2$ where if $C_2 = \langle s \rangle$ and $C_n = \langle r \rangle$, then $\varphi(s)(r) = r^{-1}$.
2. $O(2, \mathbb{R}) = SO(2, \mathbb{R}) \rtimes_{\varphi} C_2$ where if $C_2 = \langle s \rangle$ and $A \in SO(2, \mathbb{R})$, then $\varphi(s)(A) = A^{-1}$.
3. Let V be the Klein four group $V \cong C_2 \times C_2 = \langle a \rangle \times \langle b \rangle$ and let $C_3 = \langle c \rangle$. Then $A_4 \cong V \rtimes_{\varphi} C_3$ where $\varphi(c)$ is an order 3 automorphism of V .
4. Let B be the group of upper-triangular matrices in $GL(n, \mathbb{R})$. Let U be the group of upper unitriangular matrices in $GL(n, \mathbb{R})$ (that is, the group of upper triangular matrices with all 1s on the diagonal). Let T be the group of diagonal matrices in $GL(n, \mathbb{R})$. Then

$$B = U \rtimes T$$

This is called the Levi decomposition.

5. $GL(n, F) = SL(n, F) \rtimes F^*$ for any field F

Wreath products

A wreath product is a special kind of semidirect product. Let A and H be groups. Form the direct sum of $|H|$ copies of A

$$N := \bigoplus_{h \in H} A_h$$

The elements of the group N are sequences $\{a_h\}_{h \in H}$ of elements of A of which all but finitely many are the identity element of A . Then the left-action of H on itself induces an action of H on N . More precisely, a homomorphism $\varphi : H \rightarrow N$ is given on components of $\{a_h\}_{h \in H} \in N$ by

$$\varphi(h')(a_h) = a_{h'h}$$

The **wreath product** is this semidirect product $N \rtimes_{\varphi} H$, often denoted $A \wr H$.

Example

The **lamplighter group** is the wreath product $C_2 \wr \mathbb{Z}$. It is amenable.

Directed unions

Let G be a locally compact group and let $(G_i)_{i \in I}$ be a family of closed subgroups of G , partially ordered by inclusion. We say that this is a **directed family** if for every $i, j \in I$, there is a $k \in I$ so that $G_i \subseteq G_k$ and $G_j \subseteq G_k$. We say that G is the **directed union** of the family $(G_i)_{i \in I}$ if

$$\bigcup_{i \in I} G_i = G$$

Theorem

Suppose that G is the directed union of a family of closed subgroups $(G_i)_{i \in I}$. If each G_i is amenable then G is amenable.

Elementary amenable groups

Definition (Day)

The class of **elementary amenable groups** is the smallest subclass of the class of all groups that satisfies the following conditions:

- ▶ it contains all finite and all abelian groups
- ▶ if G is in the subclass and H is isomorphic to G , then H is in the subclass
- ▶ it is closed under the operations of taking subgroups, forming quotients, and forming group extensions
- ▶ it is closed under directed unions.

Examples

Solvable groups are elementary amenable, hence also nilpotent groups are elementary amenable.

Finite index amenable subgroup implies amenable

Proposition

Let G be a discrete group with a finite index subgroup H . If H is amenable then G is amenable.

For finitely generated groups we could prove this using quasi-isometries.

Proof.

It suffices to find a subgroup N of H such that $N \triangleleft G$ and N is finite index in G . Consider the action of G by left-multiplication on the cosets G/H . This gives a representation

$$\rho : G \rightarrow S_n$$

where $n = [G : H]$. Let $N = \ker(\rho)$. Then $N \triangleleft G$, and N has finite index in G since $G/N \cong \text{image}(\rho)$ which is finite. Concretely,

$$N = \bigcap_{g \in G} gHg^{-1} =: \text{core}(H)$$

Is every amenable group elementary amenable?

Day asked this question in the 1950s. The answer is **No**, but very few counter-examples are known.

In 1980 Chou showed that elementary amenable (finitely generated) groups have either polynomial or exponential growth. Grigorchuk's group of intermediate growth (1982) is amenable, so by Chou's result it is not elementary amenable. Grigorchuk's group is finitely generated but not finitely presented.

In 2005, Bartholdi and Virág showed that the “Basilica group” is amenable, which established that there are amenable groups which cannot be obtained from the class of groups of subexponential growth by the operations of group extensions and directed unions.

The so-called von Neumann conjecture

Von Neumann formulated amenability for discrete groups G using invariant measures on all subsets of G , and so proved:

Corollary

Let G be a discrete group. If G contains a free group of rank 2, then G is not amenable (as a discrete group).

Supposedly von Neumann also conjectured the following.

Conjecture

Let G be a non-amenable discrete group. Then G contains F_2 .

For finitely generated linear groups, this conjecture does hold:

Theorem (special case of Tits Alternative, 1972)

Let G be a finitely generated subgroup of $GL(n, \mathbb{R})$. Then exactly one of the following occurs:

1. G contains a free group of rank 2.
2. G contains a solvable subgroup of finite index.

Counterexamples to the von Neumann conjecture

The first counterexample to the “von Neumann Conjecture” was established by Olshanskii in 1980.

Definition

A **Tarski monster group** is an infinite group G , such that every proper nontrivial subgroup of G is a cyclic group of order a fixed prime number p .

In 1979 Olshanskii proved that uncountably many non-isomorphic Tarski monsters exist for every prime $p > 10^{75}$. In 1980 he showed that Tarski monsters are non-amenable, using a criterion due to Grigorchuk, which was based on Kesten’s random walk criterion for amenability.

Note that every nontrivial element of a Tarski monster has order p , and that Tarski monsters are finitely generated: suppose $g, h \in G$ do not commute. Then $\langle g, h \rangle$ is not cyclic of prime order, so $\langle g, h \rangle = G$.

Counterexamples to the von Neumann conjecture

In 1982 Adian showed that for $m > 1$ and $n > 665$ the group

$$G = \langle a_1, \dots, a_m \mid u^n \rangle$$

where u runs over all words in $\{a_1, \dots, a_m\}$ is not amenable, again using Grigorchuk's criterion. The group G does not contain any F_2 subgroup.

In 2001, Olshanskii and Sapir gave the first *finitely presented* counterexamples, using Adian's work (the group they construct is non-amenable because it contains G above).

There is geometry underlying these counterexamples: all are directed unions of word hyperbolic groups, but are not word-hyperbolic themselves.