The Banach–Tarski Paradox and Amenability
Lecture 25: The Ruziewicz Problem

27 October 2011

Ruziewicz Problem

Question (Ruziewicz)

Is the Lebesgue measure the only finitely additive $SO(n+1,\mathbb{R})$–invariant measure μ on the sphere S^n such that $\mu(S^n)=1$ and μ is defined on all Lebesgue subsets of S^n?

For $n=1$, Banach proved that the answer is no. The answer for $n \geq 2$ is yes. Margulis and Sullivan proved $n \geq 4$ and Drinfeld the cases $n=2,3$.

Ruziewicz Problem for $n = 1$

Proposition

Let λ be Lebesgue measure on S^1. Let A be a G_δ subset of S^1 (that is, A is a countable intersection of open sets) which is dense in S^1 and is such that $\lambda(A) = 0$. Then there exists a finitely additive invariant measure μ defined on all subsets of S^1 such that $\mu(A) = 1$.

The proof uses similar ideas to the proof that if locally compact group G satisfies the Følner Condition, then G admits an invariant mean.

Let $G = S^1$ as a *discrete* group. Then ess sup is just sup. Let

$$H = \text{span}\{g \cdot f - f \mid g \in G, f \in L^\infty(G)\}$$

and

$$Y = H \oplus \mathbb{R}\chi_G \oplus \mathbb{R}\chi_A$$

The fact that Y is a direct sum follows from $\sup_{x \in G} h(x) \geq 0$ (using the Følner Condition on S^1).

Ruziewicz Problem for $n = 1$

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Define

$$\nu(h + \alpha\chi_G + \beta\chi_A) = \alpha + \beta$$

Then ν is a bounded linear functional on Y of norm 1, so by the Hahn–Banach theorem extends to a bounded linear functional $m : L^\infty(G) \to \mathbb{R}$ of norm 1. Since $\nu(h) = m(h) = 0$ for all $h \in H$, we have that $m(g \cdot f) = f$ for all $g \in G$ and $f \in L^\infty(G)$. It is immediate that $\nu(\chi_G) = m(\chi_G) = 1$, and the fact that $m(f) \geq 0$ for $f \geq 0$ follows in the same way as in Lecture 18.

Now use m to define a finitely additive invariant measure μ on all subsets of S^1 via

$$\mu(B) = m(\chi_B)$$

Then in particular $\mu(A) = m(\chi_A) = 1$ as required.

Ruziewicz Problem for $n \geq 2$

A measure μ is absolutely continuous with respect to Lebesgue measure λ if $\mu(A) = 0$ whenever $\lambda(A) = 0$.

Proposition

Let ν be an $SO(n+1, \mathbb{R})$–invariant, finitely additive measure defined on all Lebesgue subsets of S^n, for $n \geq 2$. Then ν is absolutely continuous with respect to Lebesgue measure.

The proof uses the following version of the Banach–Tarski Paradox, which we proved in Lecture 5:

Theorem

For all $n \geq 2$, any two subsets of S^n, each of which has nonempty interior, are $SO(n+1)$–equidecomposable. In particular, S^n is equidecomposable with every subset of it whose interior is nonempty.

Ruziewicz Problem for $n \geq 2$

Let E be a subset of S^n with $\lambda(E) = 0$. We have to show $\nu(E) = 0$.

Suppose D_i is a sequence of open discs in S^n with diameters $\to 0$. Then $\nu(D_i) \to 0$, since ν is finitely additive, and n_i, the number of disjoint translates of D_i in S^n, is going to ∞. By the Theorem, S^n is equidecomposable with D_i. Thus E is equidecomposable to a subset E_i of D_i.

Since E is Lebesgue measurable and $\lambda(E) = 0$, any subset of E is measurable. It follows that the set E_i is Lebesgue measurable, since E_i is a finite union of translates of subsets of E. Thus

$$\nu(E) = \nu(E_i) \leq \nu(D_i)$$

so $\nu(E) = 0$.

Ruziewicz Problem for $n \geq 4$

The key result to show that Lebesgue measure is the only finitely additive $SO(n+1, \mathbb{R})$–invariant measure defined on all Lebesgue measurable subsets of S^n for $n \geq 4$, with total measure 1, is the following:

Theorem (Margulis, Sullivan)

For $n \geq 4$, the group $SO(n+1, \mathbb{R})$ has a finitely generated dense subgroup with Kazhdan's Property (T).

Proof.

Arithmetic subgroups. □

Ruziewicz Problem for $n \geq 4$

Proposition

Let Γ be a finitely generated subgroup of $SO(n+1, \mathbb{R})$. Then Γ acts on the Hilbert space $\mathcal{H} = L^2(S^n)$. Let ρ' be this unitary representation and let ρ be the restriction of ρ' to the subspace

$$L_0^2(S^n) = \{f \in L^2(S^n) \mid \int_{S^n} f d\lambda = 0\}$$

If ρ does not weakly contain the unit representation then the Lebesgue integral is the unique invariant mean on $L^\infty(S^n)$.

The theorem of Margulis and Sullivan provides such a Γ for $n \geq 4$. If there was another measure finitely additive $SO(n+1, \mathbb{R})$–invariant measure μ defined on all Lebesgue measurable subsets of S^n for $n \geq 4$, with total measure 1, then μ would induce a different invariant mean on $L^\infty(S^n)$.

Ruziewicz Problem for $n = 2, 3$

For $n = 2, 3$ Drinfeld's proof uses:

- Ramunujan graphs (which are expanders)
- the representation theory of $PGL_2(\mathbb{R})$ and $PGL_2(\mathbb{Q}_p)$
- the action of $PGL_2(\mathbb{R})$ on the upper half plane
- the action of $PGL_2(\mathbb{Q}_p)$ on its Bruhat–Tits tree