The Banach–Tarski Paradox and Amenability
Lecture 5: Paradoxical Decompositions of Spheres

14 August 2012

Free groups and paradoxical decompositions

Theorem

A free group of rank 2 is paradoxical.

Corollary

Suppose $F = F(a, b)$ acts freely on a set X. Then X is F–paradoxical.

Theorem (Hausdorff 1914)

There are rotations ϕ and ρ about lines through the origin in $\mathbb{R}^3$ such that ϕ and ρ generate a free rank 2 subgroup of $SO(3, \mathbb{R})$.

Corollary

For all $n \geq 3$, the groups $O(n, \mathbb{R})$ and $SO(n, \mathbb{R})$ have a free subgroup of rank 2.

Corollary

For all $n \geq 3$, the groups $O(n, \mathbb{R})$ and $SO(n, \mathbb{R})$ are paradoxical.

Paradoxical decomposition of sphere minus a countable set

Let F be the free group generated by the rotations ϕ and ρ. Each nonidentity element of F is a rotation fixing some line through the origin in $\mathbb{R}^3$. So F does not act freely on the sphere S^2: each nonidentity element of F has two fixed points in S^2.

Let D be the collection of all fixed points of nonidentity elements of F. Since F is countable, D is countable.

Lemma

Let $x \in S^2 \setminus D$ and $g \in F$. Then $g(x) \in S^2 \setminus D$.

Proof.

Suppose by contradiction that $g(x) \in D$. Then there is some nontrivial $h \in F$ fixing the point $g(x)$. So

$$hg(x) = g(x) \implies g^{-1}hg(x) = x$$

so x is a fixed point of $g^{-1}hg$, contradicting $x \in S^2 \setminus D$. $\square$

Paradoxical decomposition of sphere minus a countable set

Lemma

Let $x \in S^2 \setminus D$ and $g \in F$. Then $g(x) \in S^2 \setminus D$.

In other words, F acts on $S^2 \setminus D$.

Moreover, F acts freely on $S^2 \setminus D$, since we defined D to be the set of points of S^2 which are fixed by nonidentity elements of F.

Thus $S^2 \setminus D$ is F–paradoxical. Since $F < SO(3, \mathbb{R})$ it follows that:

Theorem (Hausdorff Paradox)

There is a countable subset D of S^2 such that $S^2 \setminus D$ is $SO(3, \mathbb{R})$–paradoxical.

Note that a countable subset of the sphere could still be dense, but the sphere itself is uncountable.

From S^2 minus a countable set to S^2

Definition

Let G be a group acting on a set X. Subsets $A, B \subseteq X$ are said to be G–equidecomposable if A and B can each be partitioned into the same finite number of respectively G–congruent pieces.

Proposition

If D is a countable subset of S^2 then S^2 and $S^2 \setminus D$ are $SO(3, \mathbb{R})$–equidecomposable.

To prove this, since D is countable and there are uncountably many lines through the origin, we can choose a line ℓ through the origin that misses every point in D.
Now let $\rho = \rho_{\ell,\theta} \in SO(3, \mathbb{R})$ be rotation about the line ℓ through angle θ.

From S^2 minus a countable set to S^2

Since D is countable, for each $n \geq 1$ there are only countably many angles θ so that $\rho^n(D)$ and D have some common point. Thus the angle θ can be chosen so that for all $n \geq 1$

$$\rho^n(D) \cap D = \emptyset$$

Let

$$\overline{D} = \bigcup_{n=0}^{\infty} \rho^n(D) = D \sqcup \rho(D) \sqcup \rho^2(D) \sqcup \cdots = D \sqcup \rho(\overline{D})$$

Then

$$S^2 = \overline{D} \sqcup (S^2 \setminus \overline{D}) \sim \rho(\overline{D}) \sqcup (S^2 \setminus \overline{D}) = S^2 \setminus D$$

so S^2 and $S^2 \setminus D$ are $SO(3, \mathbb{R})$–equidecomposable.

From S^2 minus a countable set to S^2

Theorem (Hausdorff Paradox)

There is a countable subset D of S^2 such that $S^2 \setminus D$ is $SO(3,\mathbb{R})$–paradoxical.

Proposition

If D is a countable subset of S^2 then S^2 and $S^2 \setminus D$ are $SO(3,\mathbb{R})$–equidecomposable.

Corollary (Banach–Tarski Paradox)

The sphere S^2 is $SO(3,\mathbb{R})$–paradoxical.

Paradoxical decomposition of S^n, $n \geq 2$.

Corollary

The sphere S^n is $SO(n+1, \mathbb{R})$–paradoxical for every $n \geq 2$.

Induction on n.
Assume that S^{n-1} has paradoxical decomposition $S^{n-1} = A \sqcup B$.
Define A^* to be the set of points

$$(x_1, \ldots, x_n, x_{n+1}) \in S^n \setminus \{(0, \ldots, 0, \pm 1)\}$$

such that

$$\frac{1}{\sqrt{\sum_{i=1}^{n} x_i^2}}(x_1, \ldots, x_n) \in A$$

and define $B^* \subset S^n \setminus \{(0, \ldots, 0, \pm 1)\}$ similarly.

Paradoxical decomposition of S^n, $n \geq 2$.

Check that $A^* \sqcup B^*$ is a paradoxical decomposition of $S^n \setminus \{(0, \ldots, 0, \pm 1)\}$.

The proof that

$$S^n \setminus \{(0, \ldots, 0, \pm 1)\} \text{ is } SO(n+1, \mathbb{R})\text{–equidecomposable with } S^n$$

is similar to the proof that

$$S^2 \setminus D \text{ is } SO(3, \mathbb{R})\text{–equidecomposable with } S^2.$$

Paradoxical decomposition of S^n, $n \geq 2$.

In fact for odd-dimensional spheres S^3, S^5, ... we could have used:

Theorem (Deligne–Sullivan)

For all odd $n \geq 3$, $SO(n+1, \mathbb{R})$ has a free subgroup of rank 2 which acts freely on S^n.

Corollary

For all odd $n \geq 3$, S^n is $SO(n+1, \mathbb{R})$–paradoxical.

Paradoxical decomposition of S^n, $n \geq 2$.

Theorem

For all $n \geq 2$, any two subsets of S^n, each of which has nonempty interior, are $SO(n+1)$–equidecomposable. In particular, S^n is equidecomposable with every subset of it whose interior is non-trivial.

To prove this, we extend the notion of equidecomposability.

Paradoxical decomposition of S^n, $n \geq 2$.

Definition

Let G be a group acting on a set X and let A and B be subsets of X. For positive integers n and m we say that nA is equidecomposable with mB, denoted $nA \sim mB$, if n copies of A can be decomposed and then rebuilt to form m copies of B using elements of G.

Formally, for each $1 \leq i \leq n$ there is a positive integer k_i and subsets $A_{ij} \subset A$ for $1 \leq j \leq k_i$, and for each $1 \leq i' \leq m$ there is a positive integer $k'_{i'}$ and subsets $B_{i'j'} \subset B$ for $1 \leq j' \leq k'_{i'}$ such that

- $k_1 + \cdots + k_n = k'_1 + \cdots + k'_m$
- $A = \sqcup_{j=1}^{k_i} A_{ij}$ for each i and $B = \sqcup_{j'=1}^{k'_{i'}} B_{i'j'}$ for each i'
- for each i, j there are elements $g_{ij} \in G$ so that $g_{ij}(A_{ij}) = B_{i'j'}$ for a unique i' and j'

Thus X is paradoxical means precisely that $X \sim 2X$.

Paradoxical decomposition of S^n, $n \geq 2$.

Definition

Let G be a group acting on a set X and let A and B be subsets of X. For positive integers n and m we say that nA *is equidecomposable with* mB, denoted $nA \sim mB$, if n copies of A can be decomposed and then rebuilt to form m copies of B using elements of G.

Write $nA \prec mB$ if nA is equidecomposable with $B_1 \sqcup \cdots \sqcup B_m$ where each $B_j \subseteq B$.

Theorem (Banach–Schroeder–Bernstein)

If $nA \prec mB$ and $mB \prec nA$ then $nA \sim mB$.

Paradoxical decomposition of S^n, $n \geq 2$.

Lemma

Suppose that $nA \sim nB$. Then $A \sim B$.

Definition

A graph is bipartite if its vertices can be partitioned into two sets $V_1 \sqcup V_2$ such that each edge has one endpoint in V_1 and the other in V_2.

To prove the lemma, form a bipartite graph with vertex set $A \sqcup B$. For each $1 \leq i \leq n$, draw an edge from the vertex $a \in A$ to the vertex $\varphi_i(a) := g_{ij}(a) \in B$ where $a \in A_{ij}$, and say that this edge has colour i.

Then each $a \in A$ is connected to n different vertices in B, and each $a \in A$ is the source of n edges of n different colours. Moreover each $b \in B$ is the target of n different edges, but these edges might not all have different colours.

Paradoxical decomposition of S^n, $n \geq 2$.

Definition

Let $\mathcal{G}$ be a bipartite graph with vertex set $A \sqcup B$ such that $|A| = |B|$. A perfect matching is a set of edges so that every vertex of $\mathcal{G}$ is incident to precisely one of them.

Given a subset X of $A \sqcup B$, denote by $N_{\mathcal{G}}(X)$ the set of vertices of $\mathcal{G}$ which are adjacent to X.

Theorem (Hall 1935)

A perfect matching exists if and only if for every subset $X \subset A$, $|X| \leq |N_{\mathcal{G}}(X)|$.

We may apply this theorem to our graph to get a bijection $\psi : A \to B$ such that for each $a \in A$, $\psi(a) = g_{ij}(a) = \varphi_i(a)$ for some i, j.

Paradoxical decomposition of S^n, $n \geq 2$.

Let

$$A_i = \{a \in A \mid \psi(a) = \varphi_i(a)\}$$

Thus $A_i \sim \varphi_i(A_i)$ for each $1 \leq i \leq n$. So

$$A = \bigsqcup_{i=1}^{n} A_i \sim \bigsqcup_{i=1}^{n} \varphi_i(A_i) = B$$

We have completed the proof that $nA \sim nB \implies A \sim B$.

Paradoxical decomposition of S^n, $n \geq 2$.

Theorem

For all $n \geq 2$, any two subsets of S^n, each of which has nonempty interior, are $SO(n+1)$–equidecomposable. In particular, S^n is equidecomposable with every subset of it whose interior is nonempty.

Proof.

It suffices to show that $X = S^n$ is equidecomposable with any subset A with a nonempty interior.
Since A has nonempty interior, finitely many copies of A cover S^n.
Thus $X \prec kA$ for some positive integer k.
Since $A \subseteq X$, we have $A \prec X$ and thus $kA \prec kX$.
Now $X \sim 2X$ since S^n is $SO(n+1, \mathbb{R})$–paradoxical and so by induction $X \sim kX$. Thus $kA \prec kX \sim X$ and so $kA \sim X$. The Banach–Schroeder–Bernstein Theorem then implies that $kA \sim kX$.
Hence by the lemma, $A \sim X$ as required. $\square$