The Banach–Tarski Paradox and Amenability
Lecture 7: Locally Compact Groups

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Tarski's Theorem

Theorem (Tarski)

Suppose a group G acts on a set X. Then there is a finitely additive, G–invariant measure μ on X such that $\mu(X) = 1$ and μ is defined on all subsets of X if and only if X is not G–paradoxical.

Proof.

We will only prove one direction in this course.
Suppose that X is G–paradoxical with paradoxical decomposition $X = A \sqcup B$ and that such a measure μ exists. Then since μ is finitely additive

$$1 = \mu(X) = \mu(A) + \mu(B)$$

As X is G–equidecomposable with A and μ is G–invariant, $\mu(X) = \mu(A)$. Similarly, $\mu(X) = \mu(B)$. Contradiction. Hence if such a measure μ exists, X is not G–paradoxical. □

In Part II we focus on groups G (acting on themselves) which do admit finitely additive G–invariant measures, and hence do not have paradoxical decompositions.

Definitions from point-set topology

Let X be a set. A topology on X is a collection $\mathcal{T}$ of subsets of X such that:

1. $\emptyset$ and X are in $\mathcal{T}$
2. Any union of elements of $\mathcal{T}$ is in $\mathcal{T}$
3. Any finite intersection of elements of $\mathcal{T}$ is in $\mathcal{T}$

The pair $(X, \mathcal{T})$ is called a topological space. An element U of $\mathcal{T}$ is called an open set.

Example

The collection of all subsets of X is a topology on X, called the discrete topology.

If X and Y are topological spaces a function $f : X \to Y$ is continuous if for all open sets V in Y, $f^{-1}(V)$ is open in X.

A topological space $(X, \mathcal{T})$ is Hausdorff or T_2 if for every $x_1, x_2 \in X$ with $x_1 \neq x_2$, there exist open sets U_1 and U_2 with $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$.

Topological groups

Definition

A topological group is a group G that is also a Hausdorff topological space such that the group operations $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$, for all $g, h \in G$, are continuous functions.

That is, G has compatible topological and algebraic structures.

Examples

1. The vector space $\mathbb{R}^n$ with the operation of vector addition
2. The vector space $\mathbb{C}^n$ with the operation of vector addition
3. The circle group $\mathbb{T} = \{e^{i\theta} : \theta \in \mathbb{R}\}$ with the operation of complex multiplication
4. The groups $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $O(n, \mathbb{R})$ and $SO(n, \mathbb{R})$ with the operation of matrix multiplication

Connectedness of topological groups

Definition

A topological space X is:

1. connected if X cannot be written as the disjoint union of two nonempty open subsets U and V
2. totally disconnected if its only connected components are one-point sets

This definition can be applied to topological groups.

Examples

1. $\mathbb{R}^n$, $\mathbb{C}^n$ and $\mathbb{T}$ are connected.
2. $GL(n, \mathbb{R})$ has two connected components: matrices with positive determinant and matrices with negative determinant.
3. $SL(n, \mathbb{R})$ is connected.
4. $O(n, \mathbb{R})$ has two connected components, one equal to $SO(n, \mathbb{R})$ and the other consisting of orthogonal matrices with determinant -1.

Discrete groups

Any group G can be made into a topological group by equipping G with the discrete topology. This is most natural for G finite or countable. A group with the discrete topology is sometimes called a discrete group.

Examples

1. $\mathbb{Z}$ is naturally a discrete subgroup of $\mathbb{R}$. The topology on $\mathbb{Z}$ induced by its inclusion into $\mathbb{R}$ is the discrete topology.
2. $SL(n, \mathbb{Z})$ is a naturally a discrete subgroup of $SL(n, \mathbb{R})$.

In the discrete topology, each one-point set $\{g\}$ for $g \in G$ is open, so the only connected components of G are the one-point sets. Thus G with the discrete topology is totally disconnected.

Totally disconnected topological groups

An example of a totally disconnected topological group with a topology other than the discrete topology is $SL(n, \mathbb{F}_q((t)))$ where $\mathbb{F}_q$ is the finite field of order q and $\mathbb{F}_q((t))$ is the field of formal Laurent series

$$f(t) = \sum_{n=N}^{\infty} a_n t^n \quad \text{where } N \in \mathbb{Z} \text{ and for all } n \geq N,\ a_n \in \mathbb{F}_q$$

Note that $\mathbb{F}_q((t))$ is also the field of rational functions $f(t) = \frac{P(t)}{Q(t)}$ where P and $Q \neq 0$ are polynomials with coefficients in $\mathbb{F}_q$.

Define the norm of f as above by $||f|| = e^{-N} = e^{\deg(Q) - \deg(P)}$. This gives a totally disconnected metric topology on $\mathbb{F}_q((t))$ and hence on $SL(n, \mathbb{F}_q((t)))$.

Locally compact groups

Definition

A topological space X is locally compact if for each $x \in X$, there is a compact set K and an open set U such that $x \in U \subseteq K$.

Example

$\mathbb{R}$ with its usual topology is locally compact since each $x \in \mathbb{R}$ is in an interval $(a, b) \subset [a, b]$.

Definition

A locally compact topological group or locally compact group is a topological group G which is locally compact as a topological space.

All of the examples on previous slides are locally compact groups.

With the discrete topology, any group G can be made into a locally compact group. (Again, this is most natural for finite or countable groups.)

Definitions from measure theory

Let X be a set. A σ–algebra on X is a collection $\mathcal{M}$ of subsets of X such that:

1. $\emptyset$ and X are in $\mathcal{M}$
2. $\mathcal{M}$ is closed under taking complements
3. Any countable union of elements of $\mathcal{M}$ is in $\mathcal{M}$

The pair $(X, \mathcal{M})$ is called a measure space. An element A of $\mathcal{M}$ is called a measurable set.

For any topological space X, there is a smallest σ–algebra $\mathcal{B}$ in X which contains every open set. The elements of $\mathcal{B}$ are called the Borel sets of X. Closed sets, countable unions of closed sets and countable intersections of open sets are all Borel sets. A countably additive non-zero measure μ on $\mathcal{B}$ is called a Borel measure. A Borel measure is regular if for all Borel sets A

$$\mu(A) = \sup\{\mu(F) : F \subseteq A, F \text{ is closed}\} = \inf\{\mu(G) : A \subseteq G, G \text{ is open}\}$$

Haar measure

Definition

Let G be a locally compact group. A left-invariant Haar measure on G is a regular Borel measure such that for all $g \in G$ and all Borel sets A, $\mu(gA) = \mu(A)$. That is, μ is invariant under the left-action of G on itself. Similarly we can define a right-invariant Haar measure which is invariant under the right-action of G on itself.

Theorem (Haar, Weil 1930s)

Let G be a locally compact group. Then G has a left-invariant Haar measure. A left-invariant Haar measure is unique up to scalar multiples. A similar result holds for right Haar measures.

The uniqueness statement more precisely is that if μ and μ' are left-invariant Haar measures on G, there is a constant $\lambda > 0$ such that for all Borel sets A in G,

$$\mu(A) = \lambda\mu'(A)$$

Examples of Haar measure

1. Lebesgue measure on $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{T}$, restricted to the Borel subsets, is a Haar measure which is both left- and right-invariant.
2. Counting measure on a discrete group is a Haar measure which is both left- and right-invariant.

Examples of Haar measure

1. Let
$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in GL(n, \mathbb{R}) \right\}$$
A left-invariant Haar measure on G is
$$d\mu \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \frac{1}{|a|^2} da \wedge db$$
2. Haar measure on $G = SL(2, \mathbb{R})$ is computed using the Iwasawa decomposition
$$g = kan$$
where $k \in SO(2, \mathbb{R})$, $a = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, $n = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$.

Locally compact groups have compatible algebraic, topological and measure-theoretic structures.