LATTICES IN TREES AND HIGHER DIMENSIONAL COMPLEXES

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1. Introduction

Let G be a locally compact topological group. A *lattice* in G is a discrete subgroup Γ such that $\Gamma\backslash G$ carries a finite G–invariant measure, and Γ is *uniform* or *cocompact* if $\Gamma\backslash G$ is compact. Lattices in Lie groups have been well-studied. See, for example, Raghunathan [48], and for open problems the section on "Lattices in Lie groups" in this wiki. Much less is known about lattices in other locally compact groups.

We consider lattices in the following setting. Let X be a locally finite polyhedral complex, such as a tree, a product of trees, or a (classical or nonclassical) building. Let $G = \mathrm{Aut}(X)$ be the group of automorphisms, or cellular isometries, of X. With the compact-open topology, G is naturally a locally compact group. Provided $X\backslash G$ is finite, a discrete subgroup $\Gamma \leq G$ is a lattice if and only if the series $\sum_{x\in\Gamma\backslash VX} |\Gamma_x|^{-1}$ converges (Serre [8]). Much work on lattices in $\mathrm{Aut}(X)$ has been motivated by finding similarities and differences with lattices in Lie groups. Methods of geometric group theory have so far proved useful.

2. Tree lattices

Let T be a locally finite tree. The standard reference for lattices in $G = \mathrm{Aut}(T)$ is Bass–Lubotzky [8]; see also Lubotzky's survey [41]. For uniform tree lattices many questions have been answered, but there are still some significant open problems. Much less is known about nonuniform tree lattices. For instance, there is no classification.

The main analogy, as explained by Lubotzky in [41], is with lattices in rank one Lie groups over nonarchimedean local fields, such as $\mathrm{PSL}_2(\mathbf{Q}_p)$, since the Bruhat–Tits buildings of such groups are regular or biregular trees. More recently, lattices have been constructed in rank 2 Kac–Moody groups over finite fields, which also have trees as their buildings (see Carbone–Garland [17]). Note that the definition of rank is different for Kac–Moody groups: in this context, higher rank means rank ≥ 3.

Problem 1 (Farb)**.** *There is the following analogy:* $\mathrm{PSL}_2(\mathbf{Q}_p)$ *is to* $\mathrm{Aut}(T)$ *as* $\mathrm{PSL}_2(\mathbf{R})$ *is to the group of quasi-conformal homeomorphisms* $\mathrm{QC}(D^2)$*. What are the maximal closed subgroups of* $\mathrm{Aut}(T)$ *and* $\mathrm{QC}(D^2)$*? Can we use this analogy to somehow measure the distance between lattices in* $\mathrm{PSL}_2(\mathbf{Q}_p)$*, as in Teichmüller space?*

Remark: This analogy is motivated by, for example, the fact that two non-commensurable uniform lattices in $PSL_2(\mathbf{Q}_p)$ are commensurable in $\mathrm{Aut}(T)$, and similarly with $PSL_2(\mathbf{R})$ and $QC(D^2)$.

Problem 2 (Lubotzky–Mozes)**.** *Let* Γ *be a uniform lattice in* $G = \mathrm{Aut}^+(T_k)$*, where* T_k *is the* k*–regular tree. Is the commensurator* $Comm_G(\Gamma)$ *simple?*

Remark: $\mathrm{Aut}^+(T_k)$ is the subgroup of $\mathrm{Aut}(T_k)$ generated by vertex stabilizers, which is here the group of type-preserving automorphisms. Recall that two subgroups Γ and Γ' of G are *commensurable* if $\Gamma \cap \Gamma'$ has finite index in both Γ and Γ'. The *commensurator* of $\Gamma \leq G$ is the group

$$\mathrm{Comm}_G(\Gamma) = \{g \in G \mid \Gamma \text{ and } g\Gamma g^{-1} \text{ are commensurable}\}.$$

The group $\mathrm{Comm}_G(\Gamma)$ is well-defined up to conjugacy, since all uniform lattices in $G = \mathrm{Aut}^+(T_k)$ are, up to conjugacy, commensurable (Leighton [39]). This problem asks if $\mathrm{Comm}_G(\Gamma)$ is simple as a group (that is, has no nontrivial normal subgroups).

The analogy is with the commensurator of an arithmetic lattice being a simple group.

This problem has been open for some time. The answer is probably similar for a biregular tree. One could ask: when is the commensurator of a nonuniform tree lattice simple?

Problem 3 (Lubotzky–Mozes)**.** *Let Γ_1 and Γ_2 be uniform lattices in $G = \mathrm{Aut}(T)$. If $Comm_G(\Gamma_1) = Comm_G(\Gamma_2)$ (where we really mean equal as groups, not just conjugate), is Γ_1 commensurable to Γ_2?*

Remark: Recall that two subgroups Γ and Γ' of G are *commensurable* if $\Gamma \cap \Gamma'$ has finite index in both Γ and Γ'. The *commensurator* of $\Gamma \leq G$ is the group

$$\mathrm{Comm}_G(\Gamma) = \{g \in G \mid \Gamma \text{ and } g\Gamma g^{-1} \text{ are commensurable}\}.$$

Let $C = \mathrm{Comm}_G(\Gamma)$. The group C is well-defined up to conjugacy, since all uniform lattices in G are, up to conjugacy, commensurable (Leighton, [39]). This is then the same question as: is the normalizer $N_G(C) = C$? The answer is yes for algebraic groups. The answer is probably similar for a biregular tree. A further generalization would be to uniform trees T, that is, those where $\mathrm{Aut}(T)$ admits a uniform lattice (characterized by Bass–Kulkarni [7]).

Problem 4. *For Γ a nonuniform lattice in $G = \mathrm{Aut}(T)$, what are the possible closures of the commensurator $Comm_G(\Gamma)$ in G? Find an example where the commensurator is neither dense nor discrete. Give a criterion to ensure density or discreteness of the commensurator.*

Remark: Recall that two subgroups Γ and Γ' of G are *commensurable* if $\Gamma \cap \Gamma'$ has finite index in both Γ and Γ'. Examples due to Bass, Lubotzky and Mozes are known with dense commensurator and with discrete commensurator (see [8]), and density of the commensurator was shown to hold for many nonuniform tree lattices of so-called Nagao type by Abramenko–Rémy [1]. It is not known whether these are the only possibilities. Arithmetic lattices in connected semisimple Lie groups (with trivial center and no compact factors) are characterized by having dense commensurators (Margulis [43]). All uniform tree lattices are known to have dense commensurators (Liu [40]).

Problem 5 (Mozes)**.** *The group $\mathrm{PGL}_2(\mathbb{F}_p((t)))$ has Bruhat–Tits building the $(p+1)$–regular tree T_{p+1}. Prove that nonuniform lattices in $\mathrm{PGL}_2(\mathbb{F}_p((t)))$ have dense commensurator in $\mathrm{Aut}(T_{p+1})$.*

Remark: The group $\mathrm{PGL}_2(\mathbb{F}_p((t)))$ is of infinite index in $\mathrm{Aut}(T_{p+1})$.

Problem 6 (Goldschmidt–Sims)**.** *Let $T = T_{p,q}$ be the (p,q)–biregular tree. If p and q are prime, does $G = \mathrm{Aut}(T)$ admit only finitely many (conjugacy classes of) uniform lattices with quotient an edge?*

Remark: This question can be phrased in purely group-theoretic terms: are there only finitely many amalgams of finite groups $A *_C B$ such that $[A : C] = p$, $[B : C] = q$, and any normal subgroup of C which remains normal under the inclusion into both A and B is trivial? When $p = q = 3$, there are exactly 15 such amalgams (Goldschmidt [29]). Djokovic–Miller [21] showed that there are exactly 7 such amalgams when $p = 2$ and $q = 3$. Fan [23] showed that there are only finitely many such amalgams $A *_C B$ when the order of C is a power of a prime distinct from the primes p and q. If either p or q is not prime, there are infinitely many such amalgams, as shown by Bass–Kulkarni [7]. In higher dimensions, Glasner [28] showed that for the product of trees $T_p \times T_q$, when p and q are both prime there are (up to conjugacy) only finitely many irreducible lattices with quotient a square.

3. Lattices for products of trees

Let T_1 and T_2 be locally finite trees. Then the group $\mathrm{Aut}(T_1 \times T_2)$ is isomorphic to $\mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$. A lattice $\Gamma \leq \mathrm{Aut}(T_1 \times T_2)$ is *irreducible* if it is not commensurable to a product $\Gamma_1 \times \Gamma_2$ of lattices $\Gamma_i \leq \mathrm{Aut}(T_i)$ for $i = 1, 2$.

Problem 7 (Burger–Mozes)**.** *Let Γ be an irreducible uniform lattice in $\mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$. For $i = 1, 2$, let H_i be the closure of the projection $\pi_i : \Gamma \longrightarrow \mathrm{Aut}(T_i)$. What are the possible H_is? What about nonuniform lattices?*

Remark: We assume the projections are locally primitive, that is, that the restriction of their action to the star of any vertex of the tree T_i is primitive. If the irreducible lattice Γ is uniform then the projections cannot be dense in either of $\mathrm{Aut}(T_i)$ (Burger–Mozes [15]). If Γ has an infinite linear representation over characteristic 0, then Γ is an extension of an arithmetic lattice (Burger–Mozes–Zimmer [16]). D. Rattaggi's thesis [49] gives many interesting examples.

Problem 8 (Burger–Mozes)**.** *Study lattices in products of three or more trees.*

Remark: One would like to construct interesting lattices. Here "interesting" means, in particular, examples which are not arithmetic or which are not extensions of arithmetic lattices. It is not obvious how to make deformations with ≥ 3 trees. Maybe these examples will give new phenomena e.g. a group which is residually finite and not linear?

Problem 9 (Burger–Mozes)**.** *Define and study lattices in an infinite product of trees: "adeles".*

Remark: There is a chance that all lattices are arithmetic here.

4. Higher-dimensional complexes and their lattices

These problems are motivated by the theory of lattices in semisimple Lie groups, as well as by known results for tree lattices and for lattices in products of trees. Many questions can be asked for these specific cases as well. There is a great richness of examples of polyhedral complexes of dimension ≥ 2. These include buildings, both classical and non-classical (see Brown [14] and Ronan [50]; key non-classical examples include right-angled buildings, hyperbolic buildings, and Kac–Moody buildings). Another important example is the Davis–Moussong complex for a Coxeter group (see [47]). In dimension 2, a (k, L)–*complex* is a polygonal complex X such that the link of each vertex of X is the graph L, and each 2–cell of X is a regular k–gon. This includes X which are neither buildings nor

Davis–Moussong complexes, with L for example the Petersen graph (see Świątkowski [54]). We refer the reader to the survey of Farb–Hruska–Thomas [26] for further background, motivation and examples.

Problem 10. *Classify polyhedral complexes X (of a certain class). Does "local data" determine X?*

Remark: In general there may be uncountably many non-isomorphic (k, L)–complexes (Ballmann–Brin [3], Haglund [30]). In contrast, right-angled buildings are specified up to isomorphism by a right-angled Coxeter system (W, I) and a set of cardinalities $\{q_i\}_{i \in I}$, $q_i \geq 2$ (see for example Haglund–Paulin [36]). Some Fuchsian buildings are determined by their "holonomy", a local condition, (see Haglund [32]), but otherwise hyperbolic buildings have not been classified (Gaboriau–Paulin [27]). Świątkowski [54] classified those X which are CAT(0) trivalent polygonal complexes of *Platonic symmetry*, meaning that $\mathrm{Aut}(X)$ acts transitively on flags (vertex, edge, face). A specific case of this problem is: if large balls in two complexes X and Y are isomorphic, is X isomorphic to Y?

Problem 11. *When is* $\mathrm{Aut}(X)$ *(non)discrete?*

Remark: The lattice theory of $\mathrm{Aut}(X)$ is only interesting if $\mathrm{Aut}(X)$ is nondiscrete. When X is the Davis–Moussong complex for a Coxeter system, this problem was solved by Haglund–Paulin [35]. Świątkowski [54] solved this problem for X a CAT(0) trivalent polygonal complex of *Platonic symmetry*, meaning that $\mathrm{Aut}(X)$ acts transitively on flags (vertex, edge, face). Very little is known for other cases.

Problem 12. *Let X be a locally finite polyhedral complex, other than a product of trees, such that* $\mathrm{Aut}(X)$ *acts cocompactly. Study the existence of lattices in* $\mathrm{Aut}(X)$.

Remark: For trees, there are existence theorems for both uniform (Bass–Kulkarni [7]) and nonuniform (Bass–Carbone–Rosenberg [8]) lattices. Various non-arithmetic constructions of lattices for certain higher-dimensional X may be found in work of Ballmann–Brin [4], Benakli [9], Bourdon [11, 12], Cartwright–Mantero–Steger–Zappa [19, 20], Gaboriau–Paulin [27], Thomas [55, 57], and others. A specific case of this problem is: for (k, L)-complexes, is there an L such that for some k lattices exist, and for other k they do not?

Problem 13. *Classify lattices in* $\mathrm{Aut}(X)$ *up to conjugacy, or up to commensurability.*

Remark: There are often uncountably many commensurability classes of nonuniform tree lattices (Bass–Lubotzky [8], Farb–Hruska [24]), and similarly for right-angled buildings (Thomas [55]). There are some Fuchsian buildings such that all uniform lattices are commensurable (Haglund [33]), as is the case for trees (Leighton [39]).

Problem 14. *Rigidity: to what extent does a lattice Γ in $G = \mathrm{Aut}(X)$ determine G?*

Remark: More precisely, strong (Mostow) rigidity is the following. Let X_1 and X_2 be nonpositively curved polyhedral complexes, and let Γ_i be a lattice in $G_i = \mathrm{Aut}(X_i)$ for $i = 1, 2$. Find conditions on the X_i which guarantee that any abstract group isomorphism $\phi : \Gamma_1 \longrightarrow \Gamma_2$ extends to an isomorphism $G_1 \longrightarrow G_2$. Further, determine when any two copies of Γ_i in G_i are conjugate in G_i.

There are examples of products of trees, some lattices of which are strongly rigid, others are not. What about other nonpositively curved 2-complexes?

A harder, more general problem is quasi-isometric rigidity, that is, determining when any quasi-isometry of X is a bounded distance from an isometry (automorphism). Quasi-isometric rigidity was shown for Euclidean buildings by Kleiner–Leeb [38], for Bourdon's building $I_{p,q}$ by Bourdon–Pajot [13], and for general Fuchsian buildings by Xie [60].

Another rigidity phenomenon that might be investigated is super-rigidity à la Margulis, which asks when homomorphisms defined on a lattice Γ extend to the ambient group G. See Lubotzky–Mozes–Zimmer [42], Monod [46, 45], and Shalom [53].

Problem 15. *When does a normal subgroup theorem hold for lattices in $G = \mathrm{Aut}(X)$?*

Remark: Margulis' normal subgroup theorem (see [44]) states that if Γ is a lattice in a higher-rank, center-free, semisimple Lie group G, then every nontrivial normal subgroup of Γ has finite index. Burger–Mozes [15] proved a normal subgroup theorem for lattices in products of trees, and Bader–Shalom [2] for lattices in products of very general locally compact groups.

Problem 16 (Farb)**.** *When are nonuniform lattices finitely generated?*

Remark: If G has Kazhdan's property (T) then all lattices in G are finitely generated. For example, $\mathrm{SL}_3(\mathbf{F}_q[t])$ is finitely generated as a nonuniform lattice in $\mathrm{SL}_3(\mathbf{F}_q((t)))$. In contrast, all nonuniform tree lattices are infinitely generated (Serre [52]). For $\dim(X) \geq 2$, it is known in some cases whether or not $\mathrm{Aut}(X)$ has property (T): see for example Ballmann–Świątkowski [5], and Dymara–Januszkiewicz [22]. Some nonuniform lattices for Bourdon's building $I_{p,q}$ are not finitely generated (Farb–Hruska [25]).

Problem 17 (Farb, Mozes)**.** *Let X_1 and X_2 be infinite, locally finite 2–complexes, with $\mathrm{Aut}(X_i)$ acting cocompactly for $i = 1, 2$. If $\mathrm{Aut}(X_1)$ and $\mathrm{Aut}(X_2)$ are isomorphic as groups, is X_1 isometric/simplicially isometric/homeomorphic to X_2?*

Remark: Assumptions on the X_i could be added, for example nonpositive curvature or acyclicity. For trees, if $\mathrm{Aut}(X_1)$ is isomorphic to $\mathrm{Aut}(X_2)$, then X_1 is isometric to X_2 (Bass–Lubotzky [8]), and if the commensurators of uniform lattices C_1 and C_2 are isomorphic as groups then X_1 and X_2 are isometric (for X_i regular and biregular trees; Lubotzky–Mozes–Zimmer [42]). This question is open for products of trees, and for all other higher-dimensional complexes.

Problem 18 (Farb, Mozes)**.** *Given $k \geq 3$, a graph L and a finite (k, L)–complex Y with $\Gamma = \pi_1(Y)$, is there an algorithm to determine, or can it be decided whether, Γ is irreducible (for Y a product), residually finite, simple, linear, and so on? Find necessary and sufficient conditions for these properties.*

Remark: A particular case of this problem is where Y is the quotient of a product of trees by a torsion-free lattice Γ acting freely, so that $k = 4$ and L is complete bipartite. Rattaggi [49] has examples where Γ is simple and Y is finite or infinite, and sufficient conditions for Γ to be irreducible, residually finite and simple are known.

Problem 19 (de Cornulier, Shalom)**.** *Let X be the Bruhat–Tits building associated to a higher-rank algebraic group G over a function field. Let Γ be a lattice in $\mathrm{Aut}(X)$. Is Γ arithmetic? Is Γ residually finite (asked by Burger–Mozes, Wise) ?*

Remark: Note that the algebraic group G is cocompact in $\mathrm{Aut}(X)$, but is not of finite index (Tits [58]).

Problem 20 (Farb). *Characterize lattices coming from algebraic groups.*

Remark: More precisely, let X be the Bruhat–Tits building associated to an algebraic group G over a local field. Characterize or recognize the lattices in $\mathrm{Aut}(X)$ which are lattices in G. A possibility is finite generation of nonuniform lattices.

Problem 21. *Which lattices in $G = \mathrm{Aut}(X)$ have dense commensurators in G? Is this equivalent to a lattice having infinite index in its commensurator?*

Remark: Margulis [44] characterized arithmetic lattices among lattices in semisimple Lie groups as those with dense commensurators, and showed that this property is equivalent to a lattice having infinite index in its commensurator. Haglund [31] has shown that for some Davis complexes X, the commensurator of the associated Coxeter group (which may be regarded as a uniform lattice) is dense in $G = \mathrm{Aut}(X)$. Haglund [34] and Barnhill–Thomas [6] have shown that for X a right-angled building, the commensurator of the "standard uniform lattice" is dense in G.

Problem 22. *For which X is the set of covolumes of lattices in $\mathrm{Aut}(X)$ bounded away from zero? discrete? For which X does $\mathrm{Aut}(X)$ admit towers of uniform or nonuniform lattices?*

Remark: Kazhdan–Margulis [37] showed that for G a noncompact simple real Lie group, such as $\mathrm{PSL}_n(\mathbf{R})$, the set of covolumes of lattices in G has positive lower bound, and Wang [59] proved that for such groups G, with the exceptions of $\mathrm{PSL}_2(\mathbf{R})$ and $\mathrm{PSL}_2(\mathbf{C})$, the set of covolumes is discrete. The strong finiteness theorem of Borel–Prasad [10] implies the same results for covolumes of lattices in higher-rank nonarchimedean Lie groups. Dramatically different results hold for tree lattices: see work of Bass–Kulkarni [7], Bass–Lubotzky [8], Rosenberg [51] and Farb–Hruska [24]. Thomas has shown that covolumes of lattices for right-angled buildings are very similar to those for trees [55], and that for some Fuchsian buildings [57] and Davis complexes [56], the set of covolumes is not discrete.

A *tower* of lattices in G is an infinite strictly ascending sequence of lattices in G. If G admits a tower of lattices then the set of covolumes of lattices in G does *not* have positive lower bound. Towers of uniform tree lattices were constructed by Bass–Kulkarni [7] and Rosenberg [51], and nonuniform towers by Carbone–Rosenberg [18]. Thomas has shown that for X a right-angled building, $\mathrm{Aut}(X)$ admits towers of uniform and nonuniform lattices [55].

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