Mixing time of the Swendsen-Wang process on the complete graph

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Probability on Graphs

- Many problems in statistical mechanics are of the form:
  - Consider a sequence of finite graphs $G_n = (V_n, E_n)$ with:
    - $G_n \subset G_{n+1}$ and $|V_{n+1}| > |V_n|$  
    - E.g. complete graphs $K_n$, or tori $\mathbb{Z}_n^d$
  - Construct sample space $\Omega_n$ of combinatorial objects built from $G_n$
  - Define (up to normalization) a probability $\pi_{n,\beta}(\cdot)$ on $\Omega_n$
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### Potts model:

- $\Omega = [q]^V$ for fixed $q \in \{2, 3, 4 \ldots\}$
- $\pi(\sigma) = \frac{1}{Z} e^{-\beta H(\sigma)}$ for $\sigma \in \Omega$
  - $H(\sigma) = -\sum_{uv \in E} \delta_{\sigma_u, \sigma_v}$
  - $\beta = 1/\text{temperature}$
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- If $\beta \approx 0$ then $\pi(\cdot) \approx$ uniform on $\Omega$ ("Disorder")
- If $\beta \gg 1$ preference for $u \sim v$ to have $\sigma_u = \sigma_v$ ("Order")
- Phase transition between order and disorder at critical $\beta_c$
Markov-chain Monte Carlo

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- **How does $t_{\text{mix}}$ depend on size of $\Omega$?**
  - If $t_{\text{mix}} = O(\text{poly}(\log |\Omega|))$ we have **rapid mixing**
  - Otherwise, we have **torpid mixing**
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- Irreducible aperiodic Markov chain on $[q]^V$
- Stationary distribution is $q$-state Potts model
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Note: edge probability in $G(\sigma_t^{-1}(i), \lambda/n)$ is $\lambda/n = s^i(\sigma_t)\lambda/|\sigma_t^{-1}(i)|$
Rapid mixing for $q = 2$

Potts model on $K_n$ has **continuous** phase transition when $q = 2$
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Theorem (Cooper, Dyer, Frieze & Rue 2000)

*If $q = 2$ then $SW_n(\lambda, q)$ has mixing time*

$$t_{\text{mix}} = O(\sqrt{n})$$

*for all $\lambda \notin (\lambda_c - \delta, \lambda_c + \delta)$ with $\delta \sqrt{\log n} \to \infty$ as $n \to \infty$.***
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Theorem (Long, Nachmias, Ning, & Peres 2012)

If $q = 2$ then $SW_n(\lambda, q)$ has mixing time

$$t_{\text{mix}} = \begin{cases} 
\Theta(1) & \lambda < \lambda_c \\
\Theta(n^{1/4}) & \lambda = \lambda_c \\
\Theta(\log n) & \lambda > \lambda_c 
\end{cases}$$

Ray, Tamayo, & Klein (1989) conjectured $n^{1/4}$ at $\lambda_c$
Torpid mixing for $q \geq 3$

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**Theorem (Gore & Jerrum 1999)**

*If* $q \geq 3$ *then* $SW_n(\lambda_c, q)$ *has mixing time*

$$t_{\text{mix}} = \exp(\Omega(\sqrt{n}))$$
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**Theorem (Cuff, Ding, Louidor, Lubetzky, Peres, Sly 2012)**

*If* $q \geq 3$ *then the single-site Glauber process for the Potts model has*

$$t_{\text{mix}} = \begin{cases} 
\Theta(n \log n) & \lambda < \lambda_{o}(q) \\
\Theta(n^{4/3}) & \lambda = \lambda_{o}(q) \\
\exp(\Omega(n)) & \lambda > \lambda_{o}(q)
\end{cases}$$

*where* $\lambda_{o}(q) < \lambda_{c}(q)$, *so torpid mixing begins before transition*
Magnetization distribution

Large $n$ distribution of $s(\sigma)$ known explicitly:

$$-\frac{1}{n} \log P(s(\sigma) = a) \sim \phi_\lambda(a) - \inf_{a \in \Delta_{q-1}} \phi_\lambda(a)$$

$$\phi_\lambda(a) = \sum_{i=1}^{q} \left( a_i \log a_i - \frac{1}{2} \lambda a_i^2 \right)$$

Minima of $\phi_\lambda$ correspond either to:

- **disordered** state: $s^i = 1/q$ for all $i \in [q]
- **ordered** states: $s^i = \alpha > 1/q$ and $s^j = \frac{1-\alpha}{q-1}$ for $j \neq i$

$$\lambda_0(q) := \inf \{ \lambda \geq 0 : \text{there exist ordered local minima of } \phi_\lambda \}$$

$$\lambda_d(q) := \sup \{ \lambda \geq 0 : \text{the disordered state locally minimizes } \phi_\lambda \}$$
Complete picture for $\text{SW}_n(\lambda, q)$ with $q \geq 3$

Theorem (Lin & G. 2013)

If $q \geq 3$ then $\text{SW}_n(\lambda, q)$ has mixing time

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\Theta(1) & \lambda < \lambda_0(q) \\
\Theta(n^{1/3}) & \lambda = \lambda_0(q) \\
\exp(\Omega(\sqrt{n})) & \lambda_0(q) < \lambda < \lambda_d(q) \\
\Theta(\log(n)) & \lambda \geq \lambda_d(q)
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- Gore & Jerrum’s torpid mixing result extends to a non-trivial interval $(\lambda_0(q), \lambda_d(q))$ containing $\lambda_c(q)$
- Nothing special happens at $\lambda_c(q)$
- Non-trivial scaling arises at $\lambda_0(q)$
- Low and high temperature same as Ising case
Sketch of Proof

If $Y_{t+1} := s_{t+1}^1 - \mathbb{E}[s_{t+1}^1 | \sigma_t]$ then

$$s_{t+1}^1 \approx s_t^1 + D(s_t^1) + Y_{t+1} \quad (\ast)$$

where

$$D_{\lambda,q}(x) := \theta(\lambda x)(1 - 1/q)x + 1/q - x$$

- $\theta(\lambda) n = \mathbb{E}\text{(size of giant component)}$ in Erdös-Renyi $G(n, \lambda/n)$
- $(Y_t)_{t\geq 0}$ is a sequence of martingale increments
- $\text{var}(Y_t | \sigma_t) = \Theta(n^{-1})$
- Conditioning on a certain a.a.s. event makes (\ast) exact
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- $\theta(\lambda) n = \mathbb{E}$(size of giant component) in Erdös-Renyi $G(n, \lambda/n)$
- $(Y_t)_{t \geq 0}$ is a sequence of martingale increments
- $\text{var}(Y_t | \sigma_t) = \Theta(n^{-1})$
- Conditioning on a certain a.a.s. event makes (*) exact

- Roots of $D_{\lambda,q}$ coincide with minima of Potts free energy $\phi_{\lambda,q}$

\[
\lambda_o = \inf\{\lambda \geq 0 : D_{\lambda,q}(x) \text{ has a root on } (1/q, 1]\}
\]

\[
\lambda_d = \sup\{\lambda \geq 0 : D_{\lambda,q}(1/q) = 0\}
\]
Sketch of Proof

- If $Y_{t+1} := s_{t+1}^1 - \mathbb{E}[s_{t+1}^1 | \sigma_t]$ then
  \[
  s_{t+1}^1 \approx s_t^1 + D(s_t^1) + Y_{t+1} \quad (\ast)
  \]

  where
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- Coupling arguments reduce mixing time to hitting time of $s_t^1$
Swendsen-Wang drift

\[ D_{\lambda,q}(x) := \theta(\lambda x)(1 - 1/q)x + 1/q - x \]

\[ t_{\text{mix}} = \begin{cases} 
\Theta(1) & \lambda < \lambda_o(q) \\
\Theta(n^{1/3}) & \lambda = \lambda_o(q) \\
\exp(\Omega(\sqrt{n})) & \lambda_o(q) < \lambda < \lambda_d(q) \\
\Theta(\log(n)) & \lambda \geq \lambda_d(q) 
\end{cases} \]
Discussion

- Our hitting-time estimates for $s^1_t$ explain exponent values in mixing times for several other Potts/Ising processes
  - Mixing time exponents depend on:
    - drift asymptotics near roots
    - decay of noise term
  - Give conjectured results for the Potts censored Glauber chain
    - construct couplings to complete proof
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Can one say anything for the Glauber chain for the Fortuin-Kasteleyn model?