1. (i) The order of G = no. of 2-subsets
$$= \binom{5}{2} = 10.$$

The vertex $\{a,b\}$ is adjacent to a vertex $\{c,d\}$ in 3 ways as $\{c,d\}$ is a 2-subset of $S \setminus \{a,b\}$
Thus, the degree of each vertex of G is 3.
By the Hand-Shaking Lemma,
$$3 \times 10 = 2\,(\text{size of } G)$$
so size of $G = 15$.

(ii) The vertices $\{1,2\}, \{3,4\}, \{5,1\}, \{2,3\}, \{4,5\}$
are the vertices of a 5-cycle: G_1
Similarly,
$\{1,3\}, \{5,2\}, \{4,1\}, \{3,5\}, \{2,4\}$
are the vertices of another 5-cycle : G_2.

N.B. An edge of G_1 is not an edge of G_2 & vice versa.

1. (iii) Non-adjacent vertices u, v are 2-subsets (of S) sharing one element; their union U has size 3 (i.e. $|U|=3$). A vertex adjacent to both u & v is a 2-subset disjoint from both. Since the 2-subsets are chosen from $\{1,2,3,4,5\}$, there is exactly one 2-subset disjoint from U.

(iv) G is simple, so G has no 1-cycle or 2-cycle.
A 3-cycle would need three pairwise-disjoint 2-subsets, which cannot occur among 5 elements.
A 4-cycle, in the absence of 3-cycles, requires non-adjacent vertices with two common neighbours (which cannot happen by part (iii)).
Finally, we exhibit a 5-cycle, $\{1,2\}, \{3,4\}, \{5,1\}, \{2,3\}, \{4,5\}$.
Thus G has girth 5.

2. (i) If f is an isomorphism from G to H, then f is a bijection of the vertex-sets of G and H preserving adjacency and non-adjacency and hence f preserves non-adjacency and adjacency in $\overline{G}$ and is an isomorphism from $\overline{G}$ to $\overline{H}$.

The same argument applies for the converse, since the complement of $\overline{G}$ is G.

(ii) By part (i), two simple graphs G and H are isomorphic iff $\overline{G}$ and $\overline{H}$ are isomorphic.

Hence it is sufficient to count the number of 2-regular simple graphs of order 7. (The complement of a 4-regular simple graph of order 7 is a 2-regular simple graph of order 7).

Every component of a 2-regular graph is a cycle.

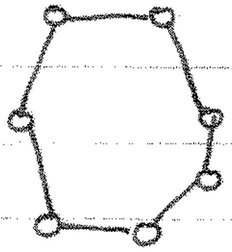
In a simple graph, each cycle has at least 3 vertices.

Thus, each different 2-regular simple graph is determined by

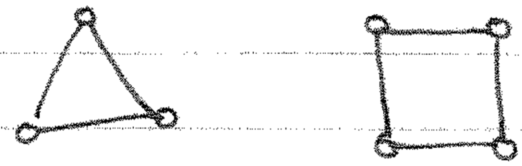
2. (ii) (cont'd)

partitioning 7 into integers of size at least 3 to be the sizes of the cycles.

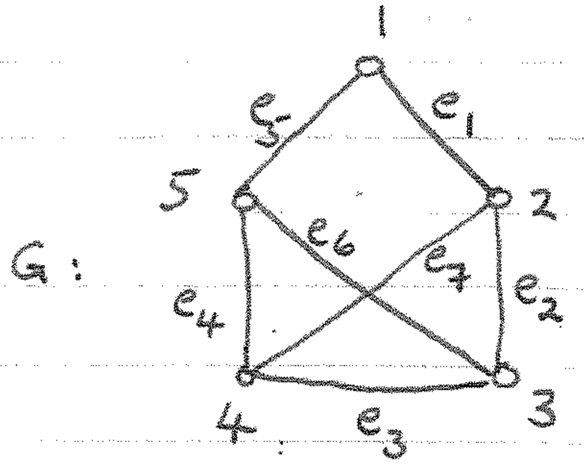
The only two graphs are exist are C_7



and C_3 C_4 (one component C_3, one component C_4)



3. (i)



$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$D(G) = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

(ii) $$M = D - A = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{bmatrix}$$

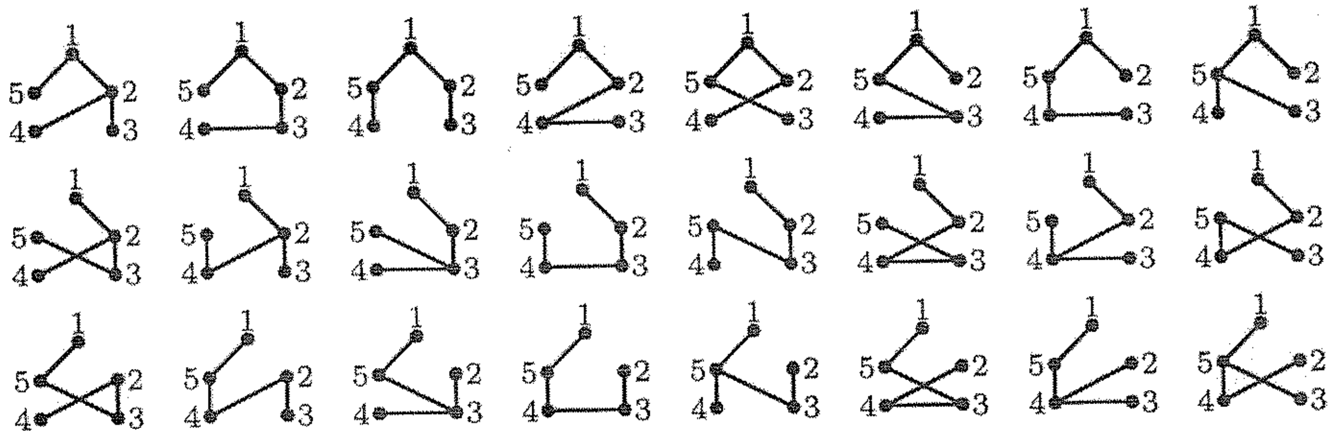
3. (ii) (cont'd)

$$\left\{\begin{matrix}\text{no. of spanning}\\ \text{trees of } G\end{matrix}\right\} = \left\{\begin{matrix}\underline{\text{any}}\text{ cofactor of}\\ M\end{matrix}\right\}$$

Taking the (1, 1) cofactor of M we get

$$(-1)^{1+1}\begin{vmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 3 \end{vmatrix} = 24.$$

The spanning trees appear below (assume in each case edges labelled as in part (i)).



4. A planar graph of order n has at most $3n-6$ edges.

Hence each planar graph G of order 11 has at most 27 edges.

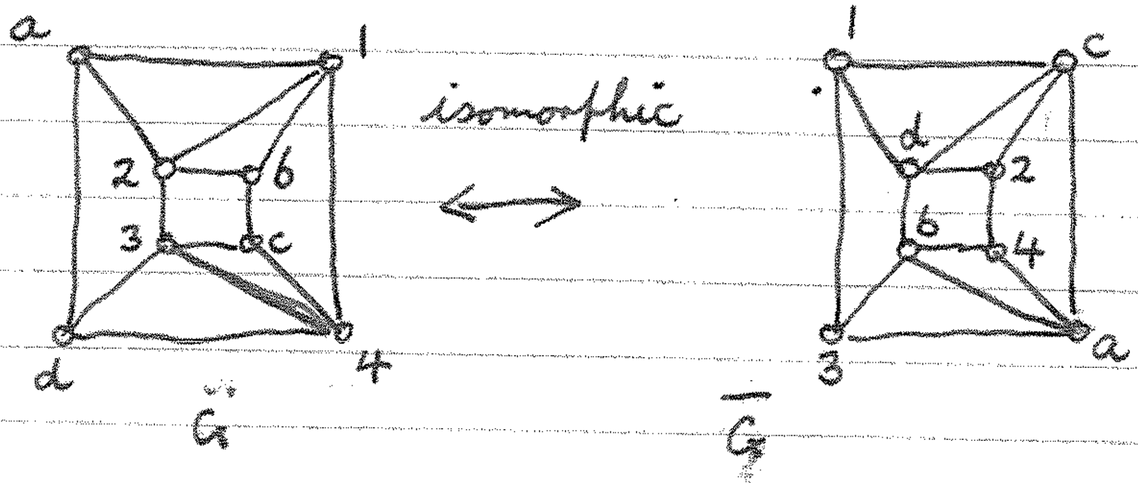
It is also a spanning subgraph of K_{11}.

K_{11} has 55 edges. Thus the complement of G has at least 28 edges and is thus non-planar.

Now consider a planar simple graph G of order >11. Any subgraph H of G of order 11 is simple and planar. From above, $\overline{H}$ is non-planar. Hence $\overline{G}$ is non-planar.

There are "many examples" of a self-complementary simple planar graph of order 8.

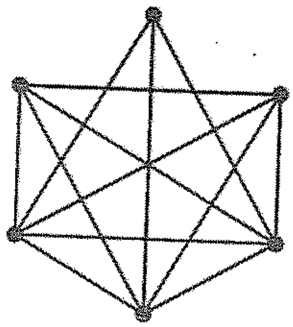
Here is one:



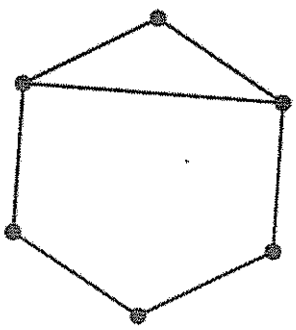
5. $$P_G(t) = P_{G+e}(t) + P_{G\backslash e}(t) \qquad (1.)$$

$$P_G(t) = P_{G-e}(t) - P_{G\backslash e}(t) \qquad (2.)$$

Let G_1 :



G_2 :

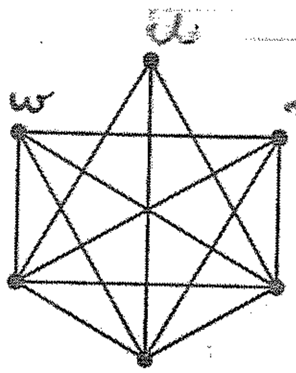


Then $P_G(t) = P_{G_1}(t) \cdot P_{G_2}(t)$.

G_1, G_2 : components of G.

We find $P_{G_1}(t)$ first.

5. (cont'd) By (1.)



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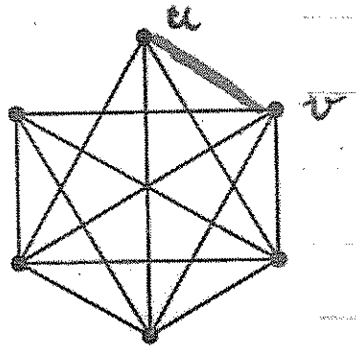

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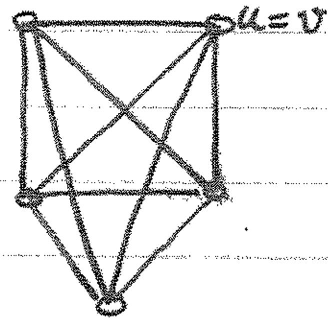

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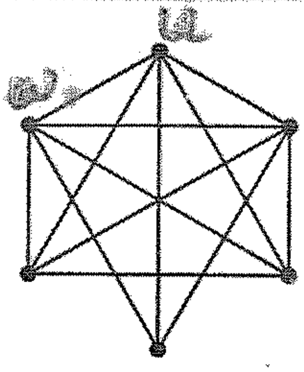

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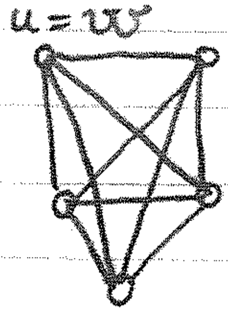

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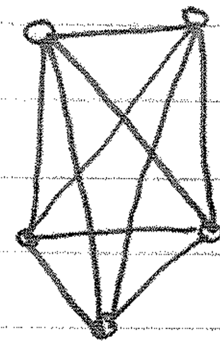


$$= K_6 + K_5 + K_5$$

$$= t(t-1)(t-2)(t-3)(t-4)(t-5)$$
$$+ 2t(t-1)(t-2)(t-3)(t-4)$$

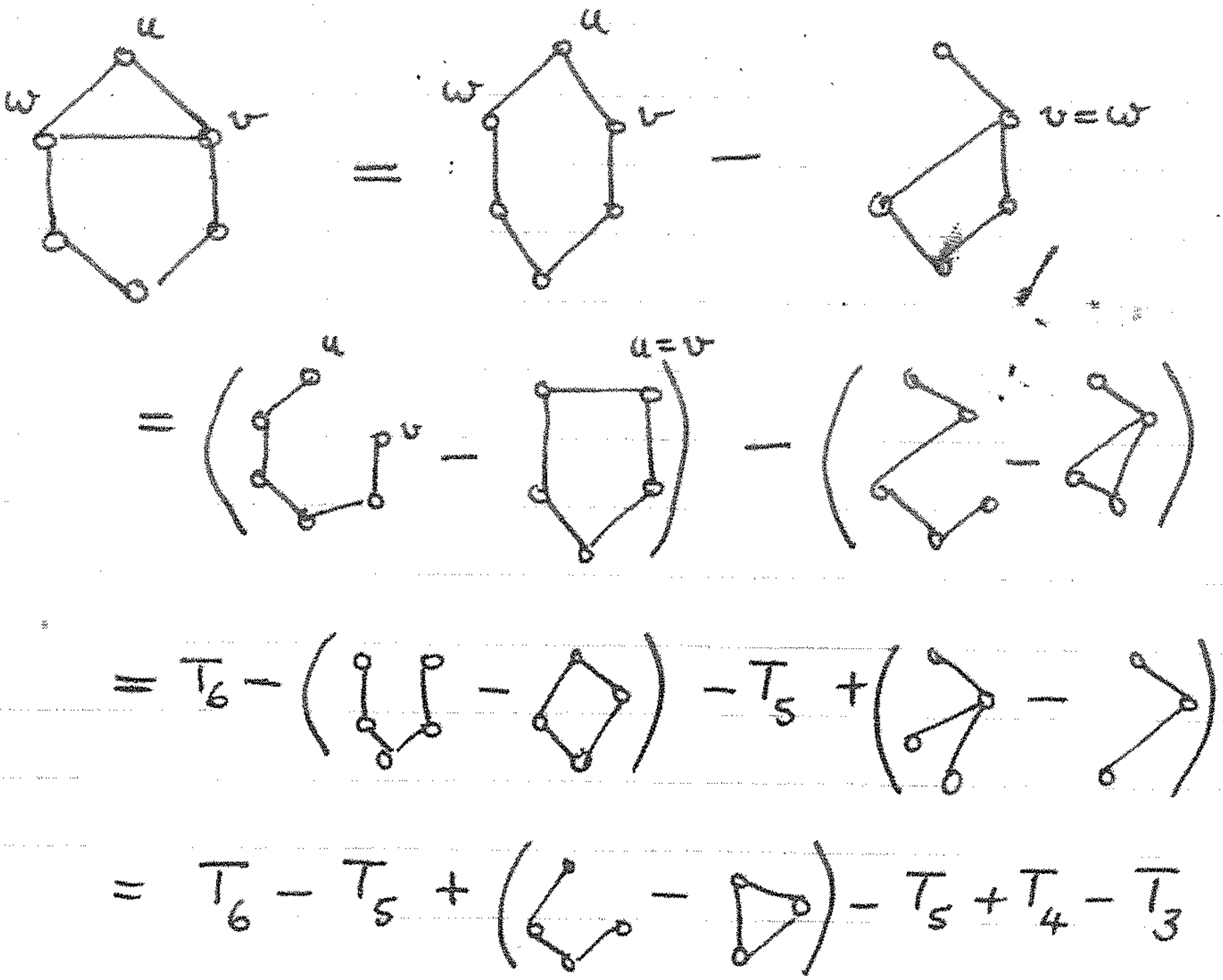
$$P_{G_1}(t) = t(t-1)(t-2)(t-3)^2(t-4)$$

5. (cont'd)

We now find $P_{G_2}(t)$.

And denote T_n : tree of order n.

We use (2.):



$$= T_6 - 2T_5 + 2T_4 - T_3 - K_3$$

$$= t(t-1)^5 - 2t(t-1)^4 + 2t(t-1)^3 - t(t-1)^2 - t(t-1)(t-2)$$

$$P_{G_2}(t) = t(t-1)(t-2)(t^3 - 4t^2 + 6t - 4) = t(t-1)(t-2)^2(t^2-2t+2)$$

So $P_G(t) = t^2(t-1)^2(t-2)^2(t-3)^2(t-4)(t^3-4t^2+6t-4)$

$$= t^{12} - 20t^{11} + 176t^{10} - 900t^9 + 2973t^8 - 6660t^7 + 10298t^6 - 10900t^5 + 7576t^4 - 3120t^3 + 576t^2.$$

Observe $P_G(0) = P_G(1) = \dots = P_G(4) = 0$,
[sub. into factored form of $P_G(t)$]
but $P_G(5) > 0$, so $\chi(G) = 5$.

This is no surprising,

$$\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$$

and

G_1 has K_5 as subgraph so
$\chi(G_1) \geq 5$, and by Brooks' Thm
$\chi(G_1) \leq 5$.
Thus $\chi(G_1) = 5$;
giving $\chi(G) = 5$.