The University of Sydney

Faculties of Arts, Economics, Education and Science

MATH2969

Discrete Mathematics & Graph Theory ADVANCED Paper 2 Graph Theory

June/July 2007 Lecturer: WD Palmer

Time Allowed: **One and a half hours**

Name: ....

SID: .... Seat Number: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 30% of the total examination;
there are 24 questions; the questions are of equal value;
all questions may be attempted.

Answers to the Multiple Choice questions must be coded onto
the Multiple Choice Answer Sheet.

The Extended Answer Section is worth 70% of the total examination;
there are 7 questions; the marks for each question are shown;
all questions may be attempted;
working must be shown.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

Extended Answer Section

Answer these questions in the answer book(s) provided.
Ask for extra books if you need them.

1. (a) [**3 Marks**] Let u and v be (not necessarily distinct) vertices of a graph G. Carefully define the terms: a uv-walk; uv-trail and a uv-path.

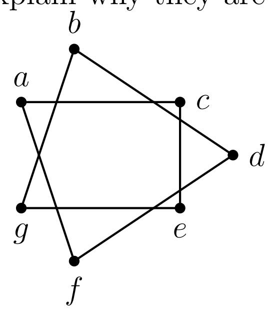
(b) [**3 Marks**] Prove that every uv-walk contains a uv-path.

(c) [**3 Marks**] Prove that if the graph G is disconnected then its complement $\overline{G}$ is connected.

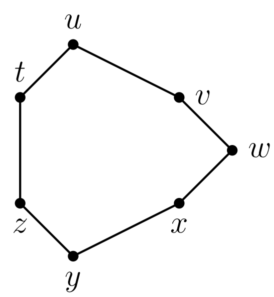
(d) [**3 Marks**] Let G and H be graphs. Show that G and H are isomorphic if and only if their complements $\overline{G}$ and $\overline{H}$ are isomorphic.

(e) [**3 Marks**] The graphs G and H, with vertex-sets $V(G)$ and $V(H)$, are drawn below. Determine whether or not G and H drawn below are isomorphic. If they are isomorphic, give a function $g : V(G) \to V(H)$ that defines the isomorphism. If they are not, explain why they are not.

G :



H :



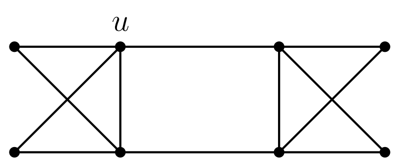
2. (a) [**2 Marks**] Show that any k-regular graph of order $(2k - 1)$ is Hamiltonian.

(b) [**1 Mark**] Copy and complete Euler's Theorem:" A connected graph is Eulerian ..."

(c) [**2 Marks**] Use Euler's Theorem to show that a connected graph with two vertices of odd degree is semi-Eulerian.

(d) [**3 Marks**]
Consider the graph G which is drawn below:

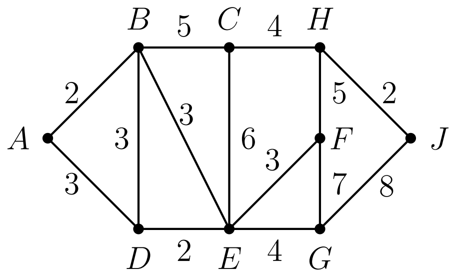
G :



Describe and apply Fleury's algorithm for finding an Eulerian circuit in G. Use vertex u as the as a starting point in your construction.

3. (a) [**3 Marks**] Let G be a simple graph of order n and with k components. Show that the size of G is at least $n - k$.

L :



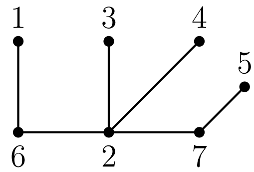
(b) [**3 Marks**] Describe Krusal's algorithm for finding a minimum weight spanning tree for an edge weighted graph. Use the Krusal's algorithm to find a minimum weight spanning tree for the weighted graoh above. Write down the weight of the minimum weight spanning tree.

(c) [**3 Marks**] Make a copy of the graph L. Use Dijstra's algorithm to add to your copy a label to each vertex, showing the least weight of all paths from A to that vertex. (In your working, show all temporary and permanent labels.)
Hence determine all least weight paths from A to J.

4. (a) [**2 Marks**] Let B be the adjacency matrix of the connected labelled graph of order n. Show that no entry of $B + B^2 + \cdots + B^{n-1}$ is zero.

(b) [**2 Marks**] Form the Prüfer sequence for the given labelled tree, T, and draw the labelled tree whose Prüfer sequence is $(3, 2, 2, 2, 4)$.

T :



(c) [**1 Marks**] State the Matrix-Tree Theorem.

(d) [**4 Marks**] Let $K_n - e$ be the graph obtained by removing an edge from the complete graph of order n. By applying the Matrix-Tree Theorem, or otherwise, show that $K_n - e$ has $(n-2)n^{n-3}$ spanning trees.

5. (a) [**1 Mark**] Define the chromatic number $\chi(G)$ of the graph G.

(b) [**1 Mark**] Draw the Petersen graph.

(c) [**3 Marks**] Let G be a simple graph with maximum vertex-degree $\Delta(G)$. Prove that G is $(\Delta + 1)$-colourable.

(d) [**1 Marks**] State Brooks' Theorem.

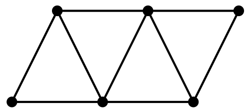
(e) [**2 Marks**] Applying Brooks' Theorem, or otherwise, to show that the chromatic number of the Petersen graph is 3.

6. (a) [**1 Mark**] Define the chromatic index (or edge chromatic number) of a graph.

(b) [**2 Marks**] State Vizing's theorem on chromatic index.

(c) [**3 Marks**] Find, supplying your reasons, the chromatic index for the graph given below:

G :



(d) [**3 Marks**] State a reduction formula for chromatic polynomials of simple graphs. Using this formula show that the chromatic polynomial, $P(t)$, of the cycle graph order n is given by $P(t) = (t-1)^n + (-1)^n(t-1)$.

7. (a) [**3 Marks**] State and prove Euler's Formula for a connected plane graph of order n, size e and with f faces.

(b) [**3 Marks**] Show that every planar graph of order less than 12 has a vertex of degree at most 4.

(c) [**3 Marks**] Let G, of order n, be a connected 3-regular plane graph in which every vertex lies on one face of length 4, one face of length 6 and one face of length 8. Determine the number of faces of G.

(d) [**3 Marks**] Show that $K_{n,n}$ is Hamiltonian if and only if $n \geq 2$. Hence show that $K_{n,n}$ has $\dfrac{n!(n-1)!}{2}$ Hamiltonian cycles.

End of Extended Answer Section