1. (a) A <u>uv-walk</u> in a graph $G=(V,E,\phi)$ is an alternating sequence
$$(u=u_0, e_1, u_1, e_2, u_2, \dots, e_k, u_k=v)$$
of vertices and edges that commences with u & ends with v.

If G is a simple graph, we frequently omit the edges $e_1,\dots,e_k$ in the sequence.

A <u>uv-trail</u> is a uv-walk in which all edges $e_1,\dots,e_k$ are distinct.

A <u>uv-path</u> is a uv-walk in which all vertices are distinct.

(b) Suppose that there is a uv-walk which is not a path (i.e. a walk containing repeated vertices). Between any 2 occurrences of a repeated vertex in the uv-walk, there is a closed walk. Remove from the uv-walk all such closed walks, and the remaining edges will form a uv-path

(c) Let v and w be vertices of G. If v and w are in different components of G, then they are adjacent in $\overline{G}$.

If v and w are in the same component of G and z is in another component, then (v,z,w) is a path in $\overline{G}$.

In either case, any two vertices can be connected by a path in $\overline{G}$. Hence $\overline{G}$ is connected.

1. (d) Let there be p vertices of degree 3.
$\quad\quad\quad\quad$ q $\quad\quad\quad\quad$ 5.
so $p+q=12$
By the handshaking lemma:

$$3p+5q = 2\times 28 = 56$$

Solving $\quad p+q=12$
and $\quad 3p+5q=56,$
we obtain $\quad p=3,\ q=10.$
Thus, the degree sequence for G is:

$$(3,3,5,5,5,5,5,5,5,5,5,5).$$

(e) Graphs G and H are isomorphic.
They are both 7-cycles.
$V(G) = \{a, b, c, d, e, f, g\}$
$V(H) = \{t, u, v, w, x, y, z\}$

We define a function $\phi: V(G) \to V(H)$
by $\phi(a) = u$
$\phi(b) = y$
$\phi(c) = v$
$\phi(d) = z$
$\phi(e) = w$
$\phi(f) = t$
$\phi(g) = x$
so that for example, edge ac in G maps to edge
$\phi(a)\phi(c) = uv$ in H.

2. (a) A connected graph is Eulerian if and only if the degree of each vertex is even.

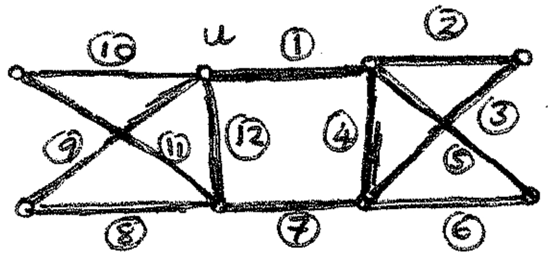
(b) The degree sequence of G is

$(2, 2, 2, 2, 4, 4, 4, 4)$.

which shows that the degree of each vertex is even. Thus, by Euler's Theorem, as G is connected, G is Eulerian.

(c) In Fleury's algorithm, we commence the Eulerian circuit at any vertex and select vertices in an arbitrary manner. As an edge is used in the construction of the circuit, it is erased. Any resultant isolated vertices are also erased.

At any vertex, use a bridge only if there is no other alternative.



3. (a) Let the order of the i th tree be n_i. The size of this tree is then $n_i - 1$. Each tree is a component of F. $\sum_{i=1}^{k} n_i = n$. Thus, the size of F is,

$$\sum_{i=1}^{k}(n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1$$

$$= n - k.$$

(b) Given a connected weighted graph of order n.

1. List the edges in order of ascending weight.

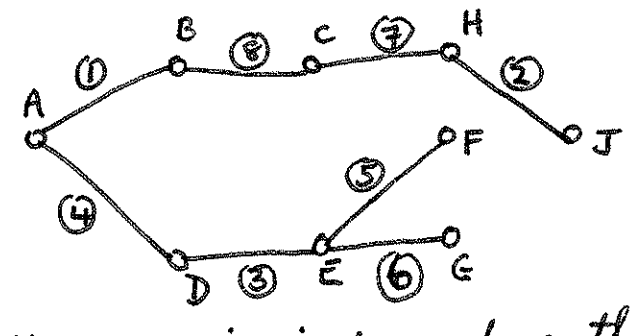
2. Construct a tree (has $n-1$ edges) by 1st drawing the vertices & no edges.

3. Add an edge of least weight.

4. Continue to add edges of least weight, without making a cycle, until we have a spanning tree.

1.

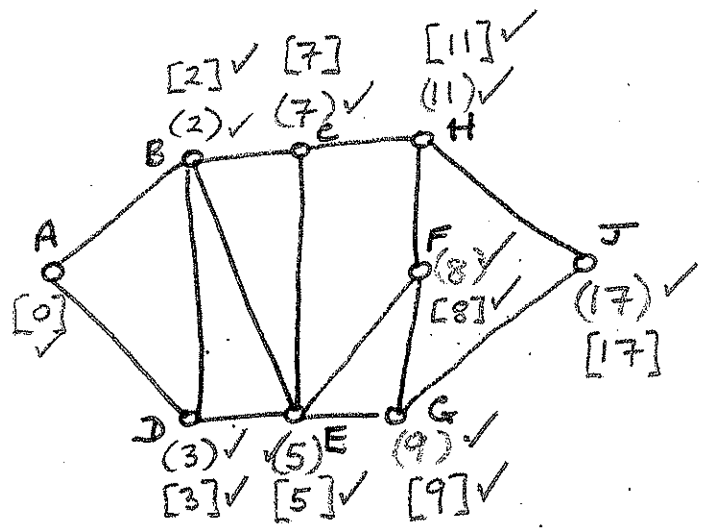
AB	HJ	DE	AD	BD	BE	EF	EG	CH	BC	HF	CE	GF	GC
2	2	2	3	3	3	3	4	4	5	5	6	7	8
✓	✓	✓	✓			✓	✓	✓	✓				



The nos. in circles show the order of construction. Weight $= 2 + 2 + 2 + 3 + 3 + 4 + 4 + 3$
$= 25.$

(c)

L :



least weight paths:

(A, B, E, G, J)

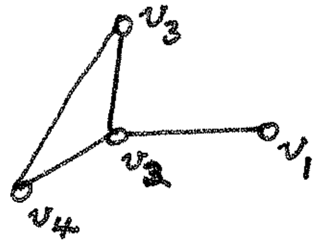
(A, D, E, G, J)

Both have weight, 17.

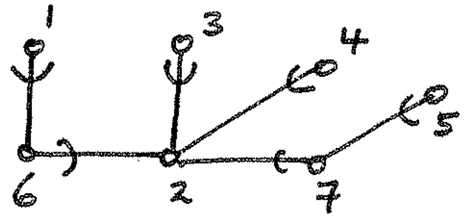
4. (a)

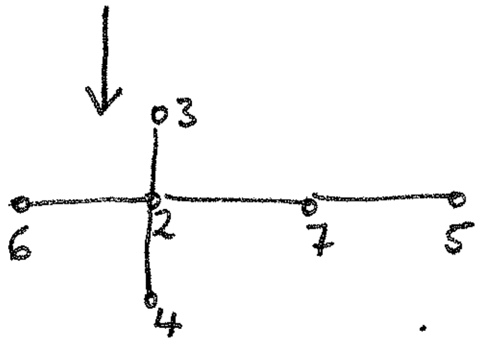
$$A = \begin{array}{c|c} & \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \hline & \left[\begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{matrix}\right] \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} \end{array}$$

$\deg(v_1) = 1$
$\deg(v_2) = 3$
$\deg(v_3) = 2$
$\deg(v_4) = 2$



(b)





(6, ..., ...)

and so on

Prüfer sequence: $(6, 2, 2, 7, 2)$

Consider the Prüfer sequence:

$(3, 2, 2, 2, 4)$

In the corresponding labelled tree; S

vertices 1, 5, 6, 7 are leaves

vertex 2 has degree 4

.. 3 2

... 4 2

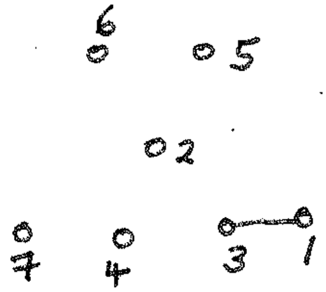
order of S = 7, size of S = 6.

4. (b) (cont'd)

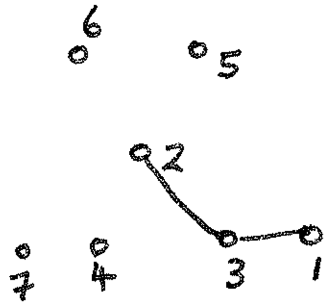
The smallest element in $\{1,2,3,4,5,6,7\}$ not appearing in $(3,2,2,2,4)$ is 1.

In S, we join 1 to 3; delete 3 from the sequence:

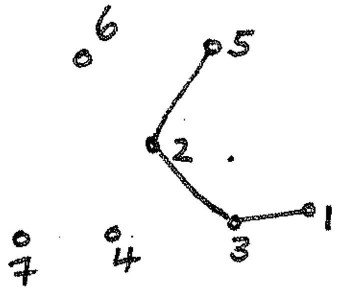
$(3,2,2,2,4)$
$\{1,2,3,4,5,6,7\}$



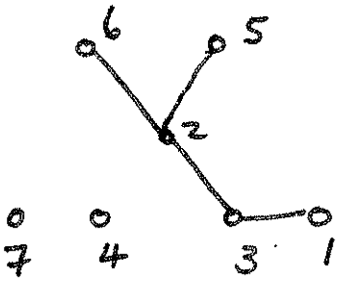
$(2,2,2,4)$
$\{2,3,4,5,6,7\}$



$(2,2,4)$
$\{2,4,5,6,7\}$

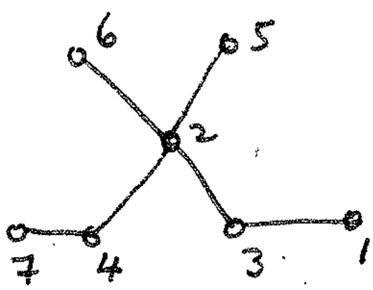


$(2,4)$
$\{2,4,6,7\}$



(4)
$\{2,4,7\}$

S:

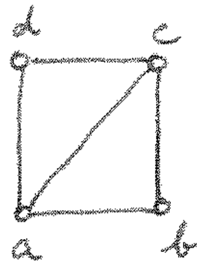


$\{2,4\}$

4.(c) Let G be a loopless labelled graph with adjacency matrix A and degree matrix D.

Then all cofactors of the matrix $M = D - A$ are equal and their common value is the no. of spanning trees of G.

(d) G:



$$D = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \end{array} \begin{array}{c} a \\ b \\ c \\ d \end{array} \qquad A = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} a \\ b \\ c \\ d \end{array}$$

$$M = D - A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

No. of spanning trees

$$= \begin{vmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix} = \text{cofactor of entry } (1,1)$$

$$= 2\begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} -1 & 3 \\ 0 & -1 \end{vmatrix}$$

$$= 2(6-1) + (-2) = 8.$$

5. (a) Let G be a graph. A <u>proper colouring</u> of G is an assignment of colours to the vertices of G s.t. no 2 adjacent vertices of G have the same colour.

The <u>chromatic number</u>, $\chi(G)$, for G is the minimum no. of colours needed to properly colour G.

(b) We prove the result for the case for G a <u>simple</u> graph (N.B. the maximum vertex degree of the underlying simple graph of G $\leq$ maximum vertex degree of G).

The proof is by induction on the order of G.

Suppose G is a simple graph of order n.

Let v be any vertex of G:

$H = G - v$ is also a simple graph of order $n-1$ & largest vertex degree at most $\Delta(G)$.

By induction hypothesis, $\chi(H) \leq \Delta + 1$.

A $(\Delta+1)$ - colouring of G is then found by colouring v with a different colour from the (at most) vertices adjacent to v. Hence $\chi(G) \leq \Delta + 1$.

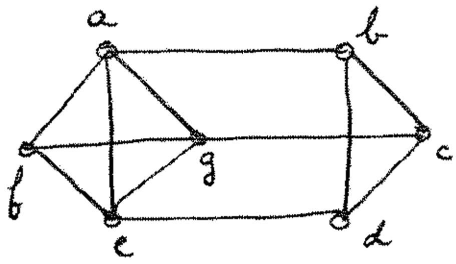
5. (c) Brooks' Theorem

Let G be a connected simple graph with largest vertex-degree $\Delta(G)$.

If G is not a complete graph or an odd cycle, then

$$\chi(G) \leq \Delta(G).$$

G:



By Brooks' Theorem (as G is not complete and not an odd cycle),

$$\chi(G) \leq 4$$

However, G has K_4 as a subgraph and $\chi(K_4) = 4$. [And $\chi(K_4) \leq \chi(G)$]

Thus $\chi(G) = 4$.

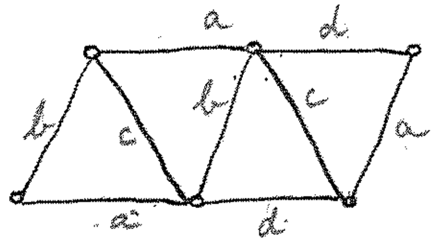
6. (a) Let G be a loopless graph of size ≥ 1.
An assignment of colours of G s.t. adjacent edges are coloured differently is a proper edge colouring of G (a k-edge colouring if k colours are used).
The minimum k for which a graph is k-edge colourable is its edge chromatic number (or chromatic index) and is denoted by $\chi'(G)$.

(b) Let G be a simple graph with largest vertex-degree Δ, then

$$\Delta \leq \chi'(G) \leq \Delta + 1.$$

(c)

G :



By Vizing's Theorem,

$$\chi'(G) = 4 \text{ or } 5.$$

However, we have exhibited a 4-edge colouring.

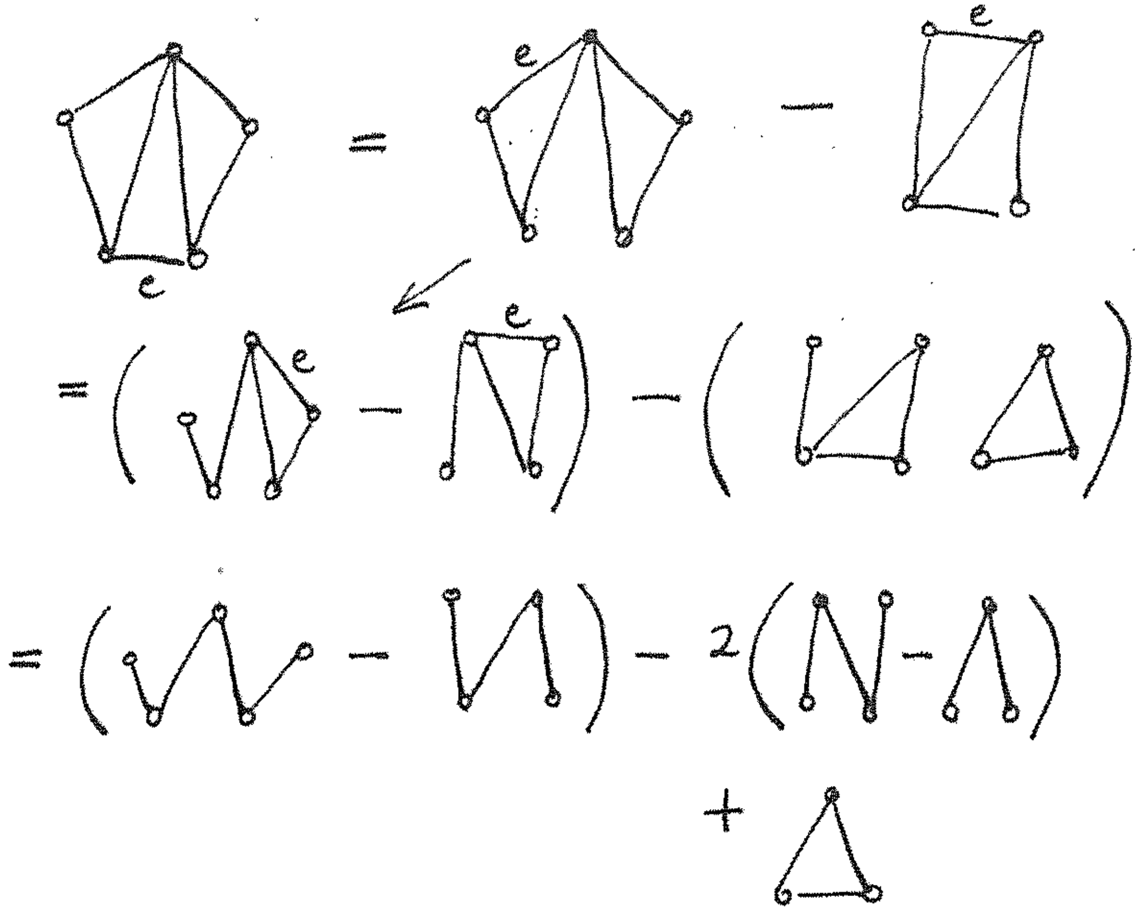
Hence $\chi'(G) = 4$.

(d) Let G be a simple graph.
Then

$$P_G(t) = P_{G-e}(t) - P_{G\backslash e}(t) \qquad e = uv \text{ exists in } G$$

$$P_G(t) = P_{G+e}(t) + P_{G\backslash e}(t) \qquad e = uv,\ u, v \text{ not adjacent in } G.$$

6. (d) (cont'd)



$$P_G(t) = t(t-1)^4 - 3t(t-1)^3 + 2t(t-1)^2 + t(t-1)(t-2)$$
$$= t^5 - 7t^4 + 18t^3 - 20t^2 + 8t.$$

7. (a) If G is a connected plane graph of order v, size e and with f faces, then
$$v - e + f = 2.$$

By induction on e, the size of G.

<u>$e=0$</u> $G = K_1$ $v = 1, e = 0, f = 1$
and so $v - e + f = 1+0+1 = 2$.

Suppose that the result is true for all connected plane graphs of size $e-1$, where $e \geq 1$.

Let us consider a connected plane graph of size e.

If G is a tree, then $v = e+1$, $f=1$ so $v-e+f = 2$ as required.

If G is not a tree, then G has a cycle. Let e be a cycle edge of G

Now $G - e$ is a connected plane graph, of order v, size $e-1$, with $f-1$ faces.
So, by the induction hypothesis:
$$v - (e-1) + (f-1) = 2,$$
which implies $v - e + f = 2$.

7(b). Since each vertex is incident to 1 face of length 4, 1 face of length 6, 1 face of length 8, there are n incidence of vertices with faces of each length.

Since every face of length l is incident with l vertices, there are $\frac{n}{4}, \frac{n}{6}, \frac{n}{8}$ faces of lengths 4, 6, 8 resp.

Hence there are $n\left(\frac{1}{4}+\frac{1}{6}+\frac{1}{8}\right)$ faces.

As G is 3-regular, it is of size $\frac{3n}{2}$.

By Euler's formula,

$$n - \frac{3n}{2} + \frac{13n}{24} = 2.$$

Thus $n = 48$.

The no. of faces is then $48 \times \frac{13}{24} = 26$.

(c) Let $G = (V, E)$ of order n, size m be self-complementary. Then $G \simeq \overline{G}$.

Every edge in the complete graph with V as the vertex-set, is either an edge of G or $\overline{G}$. Thus $\binom{n}{2} = m + m$ ($\overline{G}$ has same order, size of G)

$= n(n$

Thus $n(n-1) = 4k$, for some positive integer k.

Thus $n = 4k$ or $4k+1$.

X

7.(d) $K_{1,1}$ has no cycle. Hence it is not Hamiltonian.

For $n \geq 2$, we can list the vertices alternately from the two partite sets. Consecutive vertices are adjacent and the last is adjacent to the first, so we obtain a Hamiltonian cycle.

Specifying the order in which the vertices of each partite set will be visited determines a cycle starting a given vertex x.

Since there are n vertices in the other partite set and $(n-1)$ remaining to be visited in the same patite set as x, there are $n!(n-1)!$ ways to specify these orderings. This counts each cycle twice, as each cycle can be followed in either direction from x.

Thus the no. of Hamiltonian cycles is

$$\frac{n!\,(n-1)!}{2}.$$