1. (d) If f is an isomorphism from G to H, then f is a vertex bijection preserving adjacency and non-adjacency, and hence f preserves non-adjacency and adjacency in $\bar{G}$ and is an isomorphism from $\bar{G}$ to $\bar{H}$. The same argument applies for the converse, since the complement of $\bar{G}$ is G.

(e) Graphs G and H are isomorphic. They are both 7-cycles.

$V(G) = \{a, b, c, d, e, f, g\}$

$V(H) = \{t, u, v, w, x, y, z\}$

We define a function $\phi : V(G) \to V(H)$

by $\phi(a) = u$

$\phi(b) = y$

$\phi(c) = v$

$\phi(d) = z$

$\phi(e) = w$

$\phi(f) = t$

$\phi(g) = x$

so that for example edge ac in G maps to edge $\phi(a)\phi(c) = uv$ in H.

1. (a) A uv-**walk** in a graph $G=(V,E,\phi)$ is an alternating sequence

$$(u=u_0, e_1, u_1, e_2, u_2, \ldots, e_k, u_k=v)$$

of vertices and edges that commences with u + ends with v.

If G is a simple graph, we frequently omit the edges $e_1, \ldots, e_k$ in the sequence.

A uv-**trail** is a uv-walk in which all edges $e_1, \ldots, e_k$ are distinct.

A uv-**path** is a uv-walk in which all vertices are distinct.

(b.) Suppose that there is a uv-walk which is not a path (i.e. a walk containing repeated vertices). Between any 2 occurrences of a repeated vertex in the uv-walk, there is a closed walk. Remove from the uv-walk all such closed walks, and the remaining edges will form a uv-path.

(c) Let v and w be vertices of G. If v and w are in different components of G, then they are adjacent in $\overline{G}$.

If v and w are in the same component of G and z is in another component, then (v, z, w) is a path in $\overline{G}$.

In either case, any two vertices can be connected by a path in $\overline{G}$. Hence $\overline{G}$ is connected.

2.(a) The degree of each vertex $= k$.

$$k \geq \frac{2k-1}{2} \quad \left(= \frac{\text{order of graph}}{2}\right)$$

so, by Dirac's Theorem, the graph is Hamiltonian (when $2k-1 \geq 3$ i.e. $k \geq 2$)

There is no 1-regular graph of order $2(1)-1=1$.

(b) A connected graph is Eulerian if and only if the degree of each vertex is even.

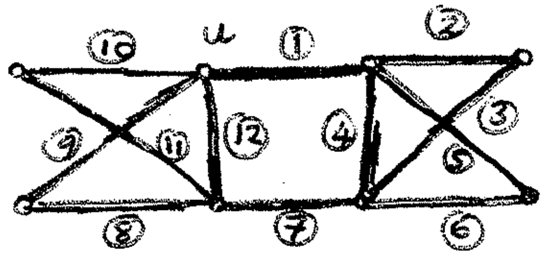
(c) Let the two odd vertices be u and v of graph G. Form the edge $e = uv$.

Each vertex in $G+e$ is even. Hence $G+e$ is Eulerian. Thus there exists an Eulerian circuit in $G+e$.

Now remove the edge, $e = uv$, to leave an Eulerian trail with endvertices u and v.

(d) In Fleury's algorithm, we commence the Eulerian circuit at any vertex and select vertices in an arbitrary manner. As an edge is used in the construction of the circuit, it is erased. Any resultant isolated vertices are also erased.

At any vertex, use a bridge only if there is no other alternative.



2007 MATH 2969 EXAM SOLUTIONS

3. (a) Proof is by mean of induction on m, the size of G.

The statement is true when $m=0$, as then $G = N_n$ (i.e. the null graph of order n).

Suppose that the assertion is true for all natural nos. $< m$.

If e is an edge of G, then $G-e$ is of order n, size $m-1$ and has $k+\delta$ components, where $\delta \in \{0,1\}$.

By the induction hypothesis, we have

$$n-(k+\delta) \leq m-1$$

and hence

$$n-k \leq m-(1-\delta) \leq m.$$

This completes the inductive argument.

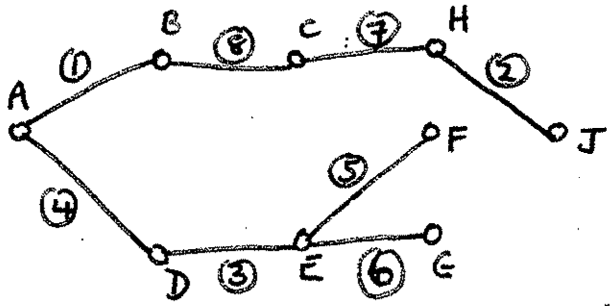
(b) Given a connected weighted graph of order n.

1. List the edges in order of ascending weight.
2. Construct a tree (has $n-1$ edges) by 1st drawing the vertices & no edges.
3. Add an edge of least weight.
4. Continue to add edges of least weight, without making a cycle, until we have a spanning tree.

3. (b) (cont'd)

1.

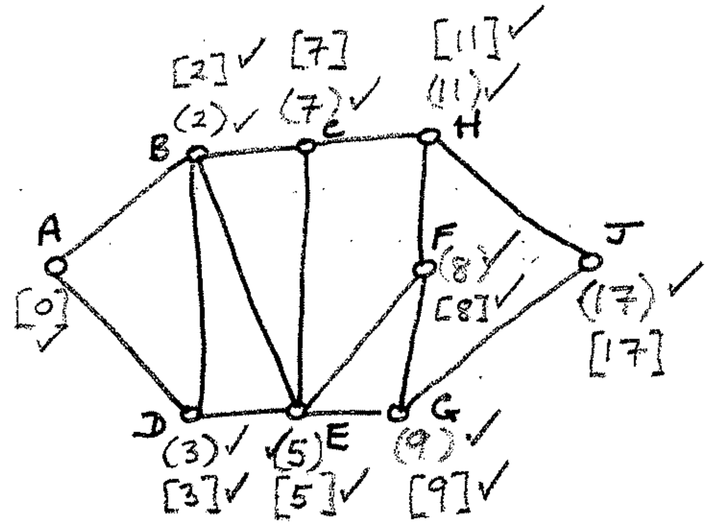
AB	HJ	DE	AD	BD	BE	EF	EG	CH	BC	HF	CE	GF	GJ
2	2	2	3	3	3	3	4	4	5	5	6	7	8
✓	✓	✓	✓			✓	✓	✓	✓				



The nos. in circles show the order of construction. Weight = 2+2+2+3+3+4+4+5
= 25.

(c)

L :



least weight paths:

(A, B, E, G, J)

(A, D, E, G, J)

Both have weight, 17.

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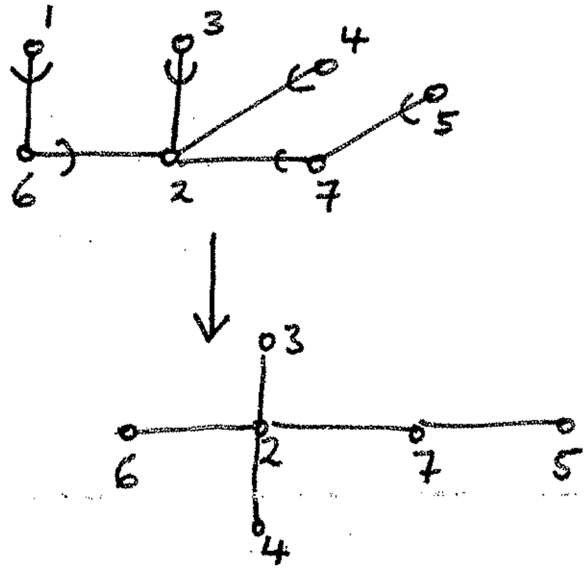
4.(a) Let $V = \{1, 2, 3, .., n\}$ be the vertex set of G. Let i, j be any 2 vertices. As G is connected, there is a path of length p between i and j, where $0 < p < n$.
Thus the (i,j) entry of B^p is non-zero.
Hence the (i,j) entry of $\sum_{k=1}^{n-1} B^k$ is non-zero.

(b)



$(6, ..., ...\ )$

and so on

Prüfer sequence: $(6, 2, 2, 7, 2)$

(c) Let G be a loopless labelled graph with adjacency matrix A and degree matrix D.
Then all cofactors of the matrix $M = D - A$ are equal and their common value is the no. of spanning trees of G.

4. (d) Label the vertices of K_n, by $v_1, \ldots, v_n$. Without loss of generality, delete the edge $e = v_{n-1}v_n$.

Then $D(K_n) = (n-1)I_n$

$$A(K_n) = \begin{array}{c c} \begin{array}{ccc} v_1 & \cdots & v_n \end{array} & \\ \begin{bmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{bmatrix} & \begin{array}{c} v_1 \\ \vdots \\ v_n \end{array} \end{array}$$

$$D(K_n - e) = \begin{bmatrix} n-1 & & & & \\ & \ddots & & 0 & \\ & & n-1 & & \\ & 0 & & n-2 & \\ & & & & n-2 \end{bmatrix}$$

(last 2 entries in diagonal $= n-2$)

$$A(K_n - e) = \begin{bmatrix} 0 & & 1 & \\ & \ddots & & \\ 1 & & 0 & 0 \\ & & 0 & 0 \end{bmatrix}$$

Thus

$$M(K_n - e) = \begin{bmatrix} n-1 & & & -1 & \\ & \ddots & & & \\ -1 & & n-1 & & \\ & & & n-2 & 0 \\ & & & 0 & n-2 \end{bmatrix}$$

4.(d)(cont'd)

The no. of spanning trees of $K_n - e$

$= $ any cofactor of $M(K_n - e)$ [Matrix-Tree Thm]

We evaluate the cofactor of the entry in position (n, n).

-1 means all entries -1

No. of spanning trees $= (-1)^{n+n} \begin{vmatrix} n-1 & & & -1 \\ & \ddots & & \\ -1 & & n-1 & \\ & & & n-2 \end{vmatrix}$ (of order $n-1$)

Now perform row, col. operations on the det:

Col. 1 = col 1 + col. 2 + ... + col (n-1)

$$= \begin{vmatrix} 1 & n-1 & & -1 \\ 1 & & \ddots & \\ \vdots & -1 & & n-1 \\ 1 & & & \\ 0 & & & n-2 \end{vmatrix}$$

all entries here -1

row 2 = row 2 − row 1

. . . .

row (n−2) = row (n−2) − row 1

$$= \begin{vmatrix} 1 & -1 & \cdots & -1 \\ & n & & 0 \\ & 0 & \ddots & \\ & & & n \\ 0 & -1 & \cdots & -1 \ (n-2) \end{vmatrix}$$

all entries here 0

(cont'd)

4.(d) expanding by the 2nd row,

$$\text{the det.} = n\begin{vmatrix} n & 0 & \cdots & 0 \\ 0 & n & \cdots & 0 \\ \vdots & & & \\ \vdots & & & \\ -1 & -1 & \cdots -1 & (n-2) \end{vmatrix} = n\,(\text{det in triangular form}$$

(of order $n-2$)

$$= n\{n^{n-2}(n-2)\}$$

product of diag. entries of triangular det.

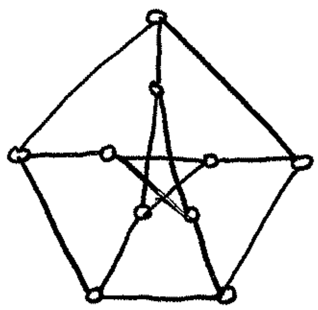
$$= n^{n-3}(n-2).$$

Thus, the no. of spanning trees in the labelled graph $K_n - e$ is $(n-2)\,n^{n-3}$.

5. (a) Let G be a graph. A proper colouring of G is an assignment of colours to the vertices of G s.t. no 2 adjacent vertices of G have the same colour.

The chromatic number, $\chi(G)$, for G is the minimum no. of colours needed to properly colour G.

(b)



(b) We prove the result for the case for G a simple graph (N.B. the maximum vertex degree of the underlying simple graph of G $\leq$ maximum vertex degree of G).

The proof is by induction on the order of G.

Suppose G is a simple graph of order n.

Let v be any vertex of G.

$H = G - v$ is also a simple graph of order $n-1$ & largest vertex degree at most $\Delta(G)$.

5. (c) (cont'd)

By induction hypothesis, $\chi(H) \leq \Delta+1$.

A $(\Delta+1)$-colouring of G is then found by colouring v with a different colour from the (at most) vertices adjacent to v. Hence $\chi(G) \leq \Delta+1$.

(d) Brooks' Theorem

Let G be a connected simple graph with largest vertex-degree $\Delta(G)$.

If G is not a complete graph or an odd cycle, then

$$\chi(G) \leq \Delta(G).$$

(e) By Brooks' Theorem, the Petersen graph, G, (which is cubic, connected and not K_4) has chromatic no. ≤ 3.

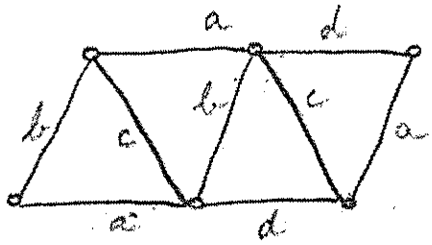
But the Petersen graph has odd cycles, so chromatic no. $\geqslant 3$.

Thus $\chi(G) = 3$.

6. (a) Let G be a loopless graph of size $\geqslant 1$.
An assignment of colours of G s.t. adjacent edges are coloured differently is a proper edge colouring of G (a k-edge colouring if k colours are used).
The minimum k for which a graph is k-edge colourable is its edge chromatic number (or <u>chromatic index</u>) and is denoted by $\chi'(G)$.

(b) Let G be a simple graph with largest vertex-degree Δ, then
$$\Delta \leq \chi'(G) \leq \Delta + 1.$$

(c)

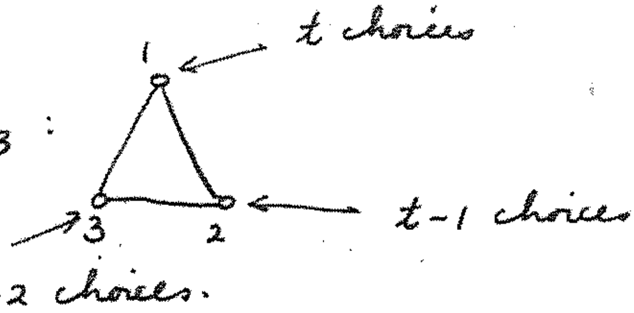
G: 

By Vizing's Theorem,
$$\chi'(G) = 4 \text{ or } 5.$$
However, we have exhibited a 4-edge colouring.
Hence $\chi'(G) = 4$.

(d) Let G be a simple graph.
G−e is graph obtained by deleting e.
G/e is graph obtained by contracting e.
Then $P_G(t) = P_{G-e}(t) - P_{G/e}(t)$.

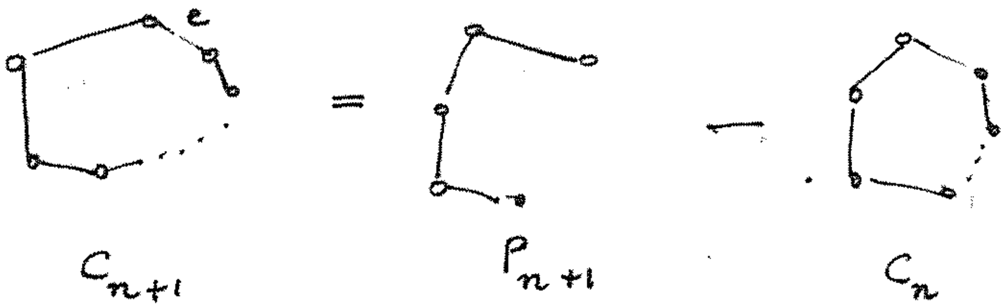
6.(d) (cont'd)

$\underline{n=3}$: C_3 :



Thus
$$\begin{aligned} P_{C_3}(t) &= t(t-1)(t-2) \\ &= (t-1)\left[t^2-2t\right] \\ &= (t-1)\left[t^2-2t+1-1\right] \\ &= (t-1)\left[(t-1)^2-1\right] \\ &= (t-1)^3+(-1)^3(t-1). \end{aligned}$$

Suppose that the chromatic polynomials for all cyclic graphs of order $i \leq n$ is given by
$$P_{C_i}(t) = (t-1)^i + (-1)^i(t-1).$$



$$\begin{aligned} P_{C_{n+1}}(t) &= t(t-1)^n - \left[(t-1)^n + (-1)^n(t-1)\right] \\ &= (t-1)(t-1)^n + (-1)^{n+1}(t-1) \\ &= (t-1)^{n+1} + (-1)^{n+1}(t-1). \end{aligned}$$

N.B. P_{n+1} is a tree of order $n+1$ so its chromatic polynomial is $t(t-1)^n$

7. (a) If G is a connected plane graph of order v, size e and with f faces, then

$$v - e + f = 2.$$

By induction on e, the size of G.

<u>$e=0$</u> $G = K_1$ $v = \phi$, $e = 0$, $f = 1$
and so $v - e + f = 1 + 0 + 1 = 2$.

Suppose that the result is true for all connected plane graphs of size $e-1$, where $e \geq 1$.

Let us consider a connected plane graph of size e.

If G is a tree, then $v = e+1$, $f = 1$ so $v - e + f = 2$ as required.

If G is not a tree, then G has a cycle. Let e be a cycle edge of G.

Now $G - e$ is a connected plane graph, of order v, size $e-1$, with $f-1$ faces.

So, by the induction hypothesis:

$$v - (e-1) + (f-1) = 2,$$

which implies $v - e + f = 2$.

7. (b) Every planar graph of order n has size $\leq 3n-6$ and hence degree sum $\leq 6n-12$. If $n < 12$, then the degree sum is at most $5n$. The pigeonhole principle implies that some vertex has degree at most 4.

(c). Since each vertex is incident to 1 face of length 4, 1 face of length 6, 1 face of length 8, there are n incidence of vertices with faces of each length.

Since every face of length l is incident with l vertices, there are $\frac{n}{4}, \frac{n}{6}, \frac{n}{8}$ faces of lengths 4, 6, 8 resp.

Hence there are $n\left(\frac{1}{4}+\frac{1}{6}+\frac{1}{8}\right)$ faces.

As G is 3-regular, it is of size $\frac{3n}{2}$. By Euler's formula,

$$n - \frac{3n}{2} + \frac{13n}{24} = 2.$$

Thus $n = 48$.

The no. of faces is then $48 \times \frac{13}{24} = 26$.

7.(d) $K_{1,1}$ has no cycle. Hence it is not Hamiltonian.

For $n \geq 2$, we can list the vertices alternately from the two partite sets. Consecutive vertices are adjacent and the last is adjacent to the first, so we obtain a Hamiltonian cycle.

Specifying the order in which the vertices of each partite set will be visited determines a cycle starting a given vertex x.

Since there are n vertices in the other partite set and $(n-1)$ remaining to be visited in the same partite set as x, there are $n!(n-1)!$ ways to specify these orderings. This counts each cycle twice, as each cycle can be followed in either direction from x.

$\therefore$ Thus the no. of Hamiltonian cycles is

$$\frac{n!\,(n-1)!}{2}.$$