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EXAM.

1. (i) yes. distinct edges, distinct vertices.

(ii) no. ae is not an edge for example.

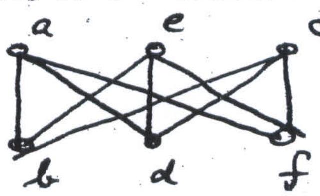
(iii) yes. degree of ea. vertex = 3.

(iv) (3, 3, 3, 3, 3, 3).

(v) $\sum_{v \in V} \deg(v) = 2e$; e = no. of edges

accept $\sum_{v \in V} \deg(v)$ is even.

(vi) Graph can be redrawn as

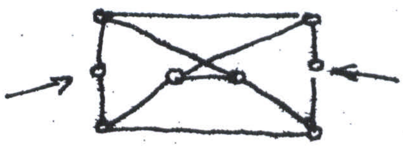


$\simeq K_{3,3}$ which is not planar.

(vii) No. All vertices are odd.

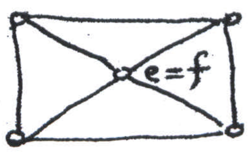
(viii) Hamiltonian cycle: e f c d a b e

(ix)



extra vertices.

(x)



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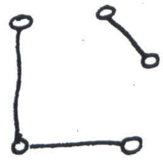
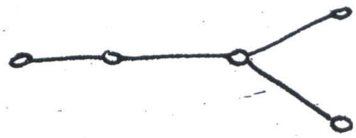
EXAM.

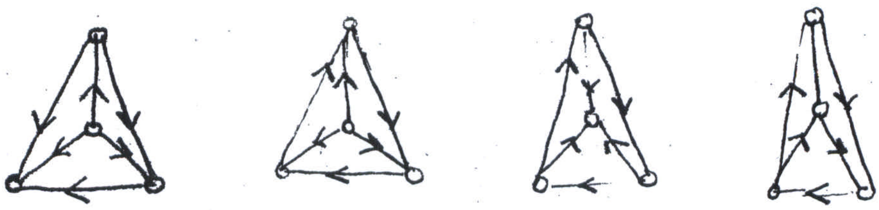
2. (i)



(ii)



(iii)

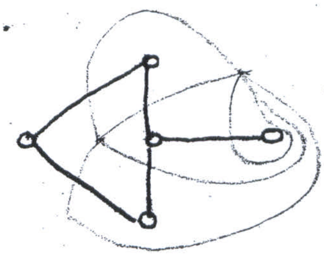


3. (i)

A graph is planar iff it contains no subgraph homeomorphic to K_5 or $K_{3,3}$

K_5 is a subgraph of K_6. Hence by Kuratowski's Thm it is not planar.

(ii)



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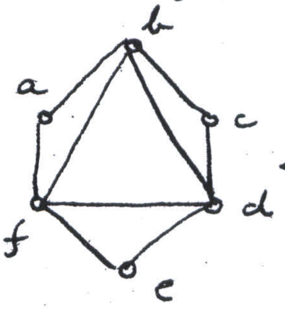
EXAM.

4 (i) yes. There exists a directed path between any two vertices. [label vertices clockwise around digraph as below:

e.g. a b c d e , c b d f a

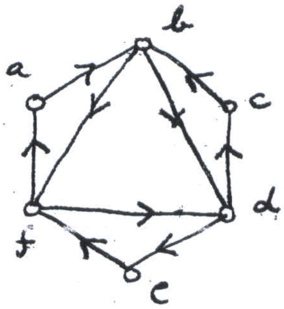
(ii)



(iii) yes. at most one arc from ea. vertex to another vertex.

(iv) yes. for ea. vertex : outdegree = indegree.

(v) no:



arcs form 3 disjoint directed cycles.

5\. (i) An Eulerian trail is a trail that contains each edge of the graph. An Eulerian circuit is a closed Eulerian trail.

We require for Eulerian circuit all d_i to be even.

For an Eulerian trail we require either all d_i even or exactly one pair of d_i to be odd (in which case the trail starts + ends at these odd vertices).

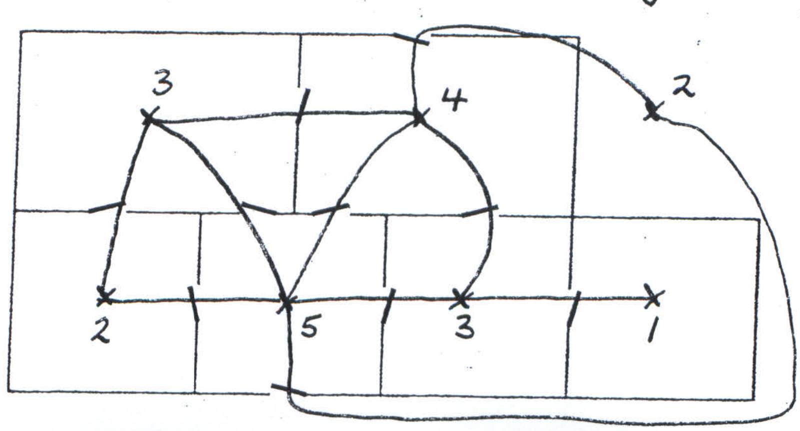
(ii) Start at a vertex (for Eulerian trail an odd vertex) and traverse the edges in an arbitrary manner:

(1.) erase edges as traversed, and isolated vertices arising from this;

(2.) at ea. stage, use a bridge if there is no alternative.

(iii) Put a vertex in ea. room and one outside of the house.

Two vertices are adjacent if we can reach one vertex from the other by passing directly through a doorway.



More than one pair of odd vertices so no Eulerian trail exists.

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EXAM.

6. (i) Prüfer sequence $= (1,2,1,4,5,2)$.

T has 8 vertices

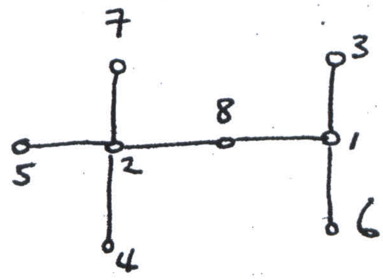
1 has degree 3
2 3
3 1
4 . . . 2
5 2
6 1
7 1
8 1

degree sequence $= (1,1,1,1,2,2,3,3)$.

(ii) Prüfer sequence $= (1,2,2,1,2,8)$.

T has 8 vertices.

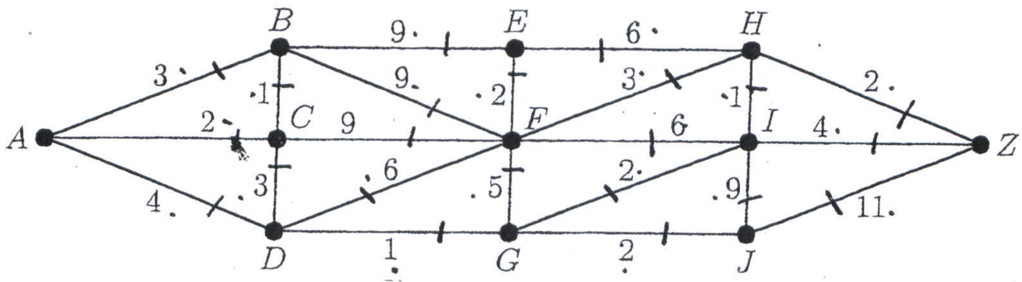
1 has degree 3
2 4
3 . . . 1
4 1
5 1
6 1
7 1
8 2



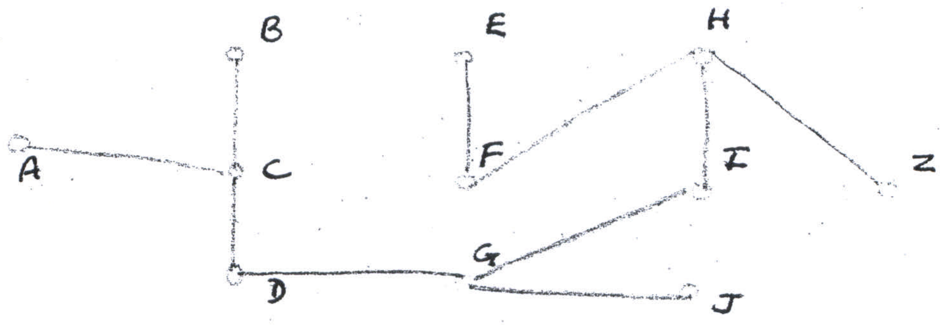
(iii) $19^{19-2} = 19^{17}$.

7

(i)
- List the edges in order of increasing weight
- Construct a tree by 1st drawing the vertices & no edges.
- Add an edge of least weight.
- Continue to add edges of least weight (without making a cycle), until we have a spanning tree.

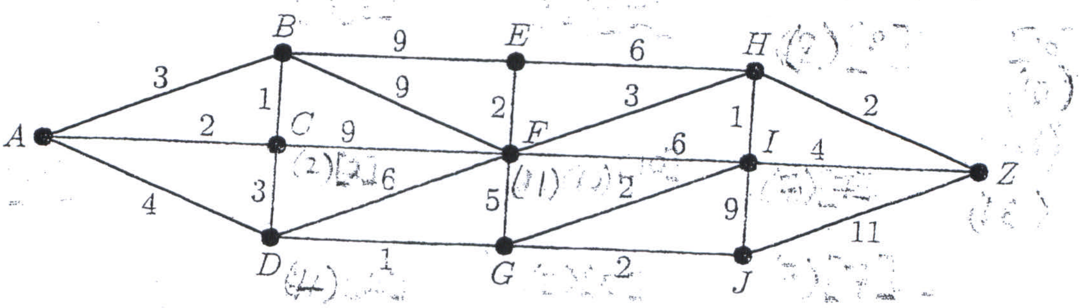


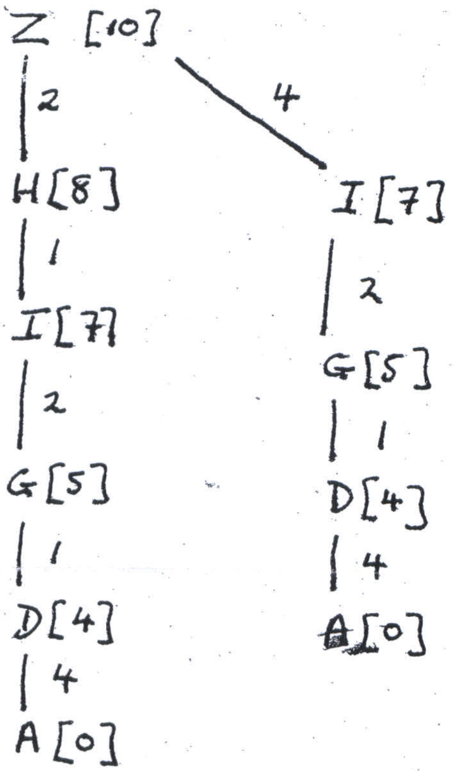
1	1	1	2	2	2	2	2	3	3	3	4	4	5	6	6	6	9	9	9	9	11
BC	DG ✓	HI ✓	AC ✓	EF ✓	GI ✓	GJ ✓	HZ ✓	AB	CD ✓	FH ✓	AD	IZ	FG	DF	EH	FI	BE	BF	CF	IJ	JZ



7.

(ii)


A
B
C
D
E
F
G
H
I
J
Z
3
2
4
1
3
9
9
9
6
1
2
5
6
3
2
6
1
9
2
4
11
2


Z [10]
2
4
H[8]
1
I[7]
2
G[5]
1
D[4]
4
A [0]
I [7]
2
G[5]
1
D[4]
4
A[0]

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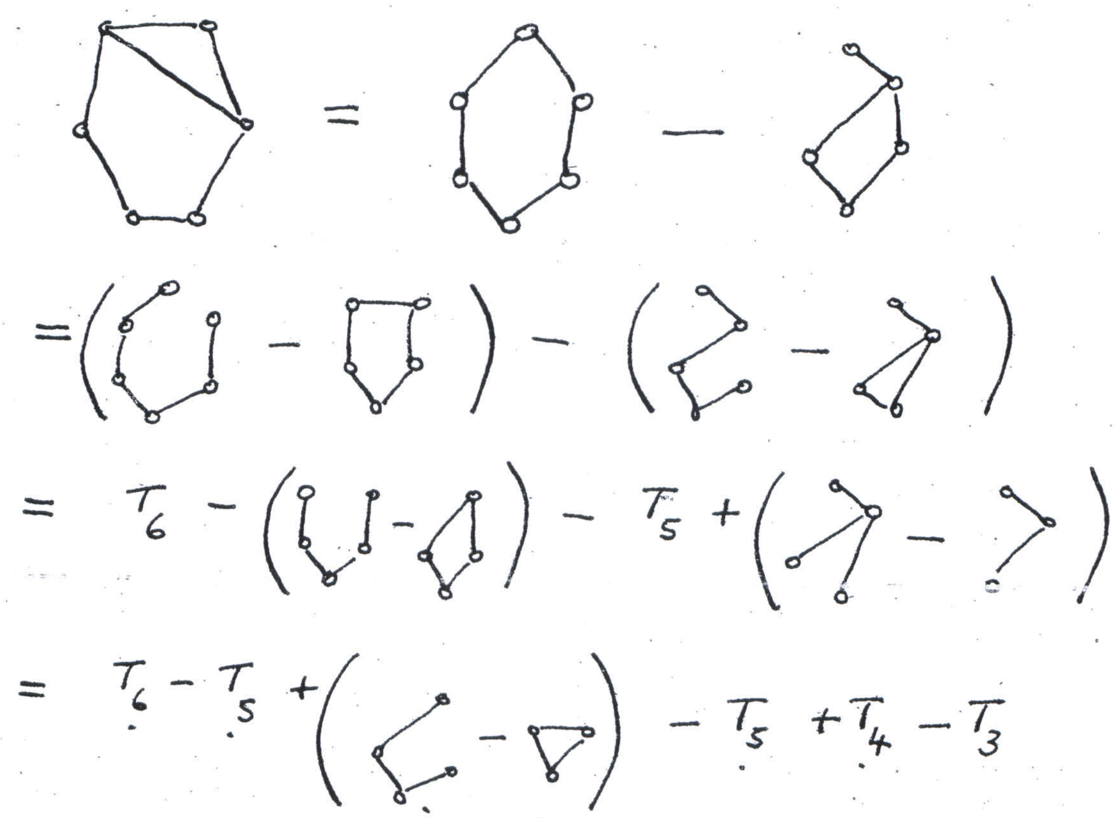
EXAM.

8

(i) $$P_G(t) = P_{G+e}(t) + P_{G\backslash e}(t)$$ $e = uv$, u, v non-adjacent in G.

$$P_G(t) = P_{G-e}(t) - P_{G\backslash e}(t)$$ $e = uv$, u, v adjacent in G.



$$= T_6 - 2T_5 + 2T_4 - T_3 - K_3$$

$$= t(t-1)^5 - 2t(t-1)^4 + 2t(t-1)^3 - t(t-1)^2 - t(t-1)(t-2)$$

$$= t(t-1)\left[(t-1)^4 - 2(t-1)^3 + 2(t-1)^2 - (t-1) - (t-2)\right]$$

$$= t(t-1)\{t-2\}^2(t^2 - 2t + 2).$$

8\. (ii) $$P_G(t) = t(t-1)(t-2)(t^2-5t+7)$$

degree of $P_G(t) = 5$, so G has 5 vertices.

Expanding $P_G(t) = t^5 - 8t^4 + 24t^3 - 31t^2 + 14t$

coeff. of $t^4 = -8$, so G has 8 edges.

9.

(i) If G is a simple graph with largest vertex-degree Δ, then $\chi'(G) = \Delta$ or $\Delta + 1$.

(ii) If two edges have the same colour they cannot be incident with a common vertex.
If K_n, there are at most $\left(\frac{n-1}{2}\right)$ edges with the same colour, n odd.
Suppose we use $n-1$ colours, we can then colour at most $(n-1) \times \frac{n-1}{2}$ edges.
K_n has $\frac{n(n-1)}{2}$ edges.
Thus $(n-1)$ colours is not sufficient to colour K_n (n odd).
Vizing's Thm says that

$$\chi'(K_n) = n-1 \text{ or } n$$

Thus, if n is odd,

$$\chi'(K_n) = n.$$

(iii) Since G is regular of degree 3, we have $\chi'(G) \geq 3$.
We obtain a 3-colouring of the edges of G by colouring the edges of a Hamiltonian cycle alternatively red + blue, then colour the remaining edges green.

10. (i)

$K_{1,n} \approx K_{n,1}$ is a tree

Conversely, if a tree is complete bipartite, then it is $K_{m,n}$ for some m and n.

But $K_{m,n}$ has no vertices of degree 1 unless m or n is 1.

(ii) K_n has n^{n-2} spanning trees each with $n-1$ edges.

Hence the total no. of all edges used in all spanning trees is $(n-1)n^{n-2}$.

Now each of the $\binom{n}{2}$ edges in K_n is equally likely to be included in a spanning tree.

Hence, the no. of spanning trees containing e is

$$\frac{(n-1)n^{n-2}}{\binom{n}{2}} = (n-1)n^{n-2} \times \frac{2}{n(n-1)}$$

$$= 2n^{n-3}.$$