MATH2069 EXAM JUNE, 2006

SOLUTIONS

1. (i) (a) yes
(b) no
(c) no

(ii) yes

(iii) no

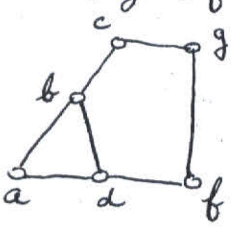
(iv) no

(v) yes

(vi) no

(vii) yes $a \to b \to c \to g \to f \to e \to d \to a$

(viii) (3,3,2,2,2,2)




(ix) 3

(x) degree sequence for G: $\begin{array}{ccccccc} e & b & d & c & g & f & a \\ (5, & 4, & 4, & 3, & 3, & 3, & 2) \end{array}$

$$\sum \text{degrees} = 5+4+4+3+3+3+2 = 24$$

no. of edges in $G = 12$.

no. of edges in $K_7 = \frac{7 \times 6}{2} = 21$.

so, no. of edges in $\overline{G} = 21 - 12 = 9$.

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2. (i) The graphs are not isomorphic.


For the graph, G: the two vertices a, b of degree 2 are not adjacent; however, for graph, H: the two vertices of degree 2 c, d are adjacent.

(ii) Hand-shaking lemma:
In any graph, the sum of the degrees of the vertices is twice the no. of edges.

In G, suppose that there is an odd no. of vertices with odd degree.
Then $\sum_{v \in G}$ degrees = an odd no.
$\neq 2 \times$ no. of edges
Hence there is a contradiction.
Thus, the no. of vertices of odd degree is even.

(iii) If G had an isolated vertex, the most no. of edges that G can have is

$$\binom{v-1}{2} = \frac{(v-1)(v-2)}{2}.$$

However, G has more than $\frac{(v-1)(v-2)}{2}$ edges.

Thus G has no isolated vertices.

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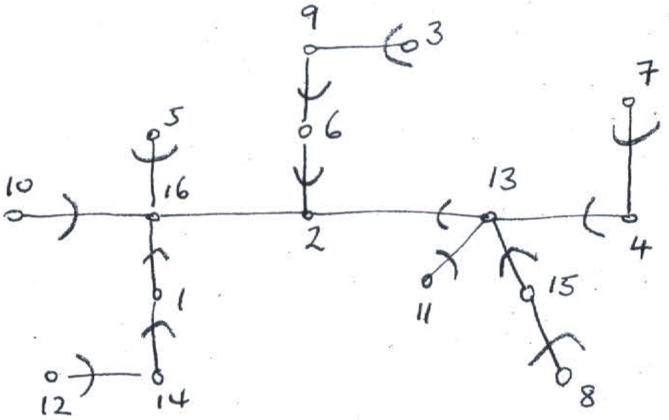
3. (i)

(a)



| vertex | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| degree | 2 | 3 | 1 | 2 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 4 | 2 | 2 | 4 |

(9, 16, 4, ~~15, 16, 13,~~ 13, 15, 6, 2, 16, 13, 14, 1, 16, 13, 2)

(b) (10, 10, 10, 3, 9, 8, 7, 1, 2).

We want a graph on 11 vertices. {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}
leaves are 4, 5, 6, 11

| vertex | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| degree | 2 | 2 | 2 | 1 | 1 | 1 | 2 | 2 | 2 | 4 | 1 |

(10, 10, 10, 3, 9, 8, 7, 1, 2)

{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}

edge (10, 4) ✗

edge (10, 5) ✗

edge (10, 6) ✗

edge (3, 10) ✗

edge (9, 3) ✗

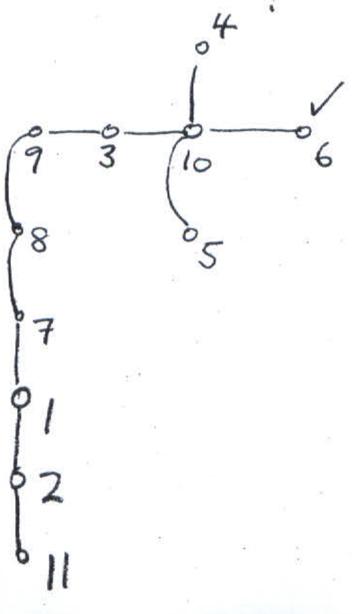
edge (8, 9) ✗

edge (7, 8) ✗

edge (1, 7) ✓

edge (2, 1)

edge (2, 11)



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3. (ii) (a)

$$A = \begin{array}{c c} & \begin{array}{cccc} a & b & g & f \end{array} \\ \begin{array}{c} a \\ b \\ g \\ f \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$\begin{aligned} \text{trace of } A^3 &= 6 \times \text{no. of triangles in } G \\ &= 6 \times 2 \\ &= 12. \end{aligned}$$

(b) Let $G$ be a connected simple labelled graph with adjacency matrix $A$ and degree matrix $D$.

Then all cofactors of $M = D - A$ are equal and their common value is the no. of spanning trees of $G$.

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$M = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

$$\text{cofactor} = (-1) \begin{vmatrix} -1 & -1 & 0 \\ 3 & -1 & -1 \\ -1 & 3 & -1 \end{vmatrix}$$

3.(ii)(b)(cont'd)

$$= (-1)\begin{vmatrix} -1 & -1 & 0 \\ 4 & -4 & 0 \\ -1 & 3 & -1 \end{vmatrix} \qquad R_2 = R_2 - R_3$$

$$= (-1)(-1)\begin{vmatrix} -1 & -1 \\ 4 & -4 \end{vmatrix}$$

$$= (-1)(-1)(4+4) = 8.$$

4. (i) Given a connected weighted graph, G, with n vertices.

(1) List the edges in order of increasing weight

(2) Construct the tree by first drawing the n vertices (with no edges)

(3) Add an edge of least weight.

(4) Continue adding edges of least weight, without making a cycle, until we have a spanning tree (i.e. have n−1 edges).

AB ✓ 8 ✓
AC ✓ 1
BD ✓ 4 ✓
BC ✓ 7 ✓
CD 2 ✓
CE 13 ✓
DE 10 ✓
DF 12 ✓
EF 2 ✓
EG 1 ✓
FG 1 ✓
FH 1
GH 3 ✓
BE 5 ✓

The algorithm is adapted by:

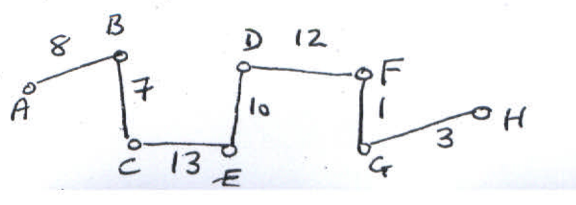
(a) starting with GF (whatever its weight is)

(b) listing the edges in decreasing weight order

(c) adding edges of most weight.

| ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | | | ✓ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| GF | CE | DF | DE | AB | BC | BE | BD | GH | CD | EF | EG | FH |
| 1 | 13 | 12 | 10 | 8 | 7 | 5 | 4 | 3 | 2 | 2 | 1 | 1 |

Eight vertices, so need 8−1 = 7 edges



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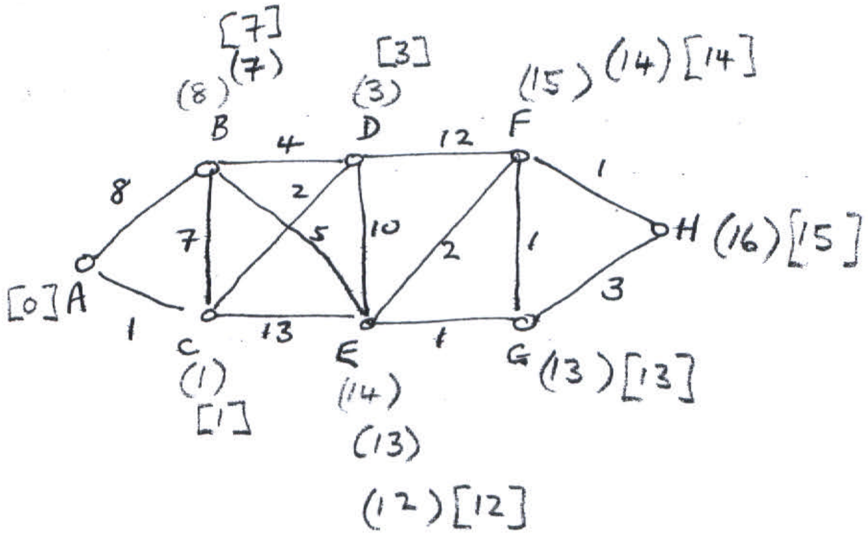
4. (ii)




$$H \xrightarrow{1} F \xrightarrow{1} G \xrightarrow{1} E \xrightarrow{5} B \xrightarrow{4} D \xrightarrow{2} C \xrightarrow{1} A$$

$$F \xrightarrow{2} E \xrightarrow{5} B \xrightarrow{4} D \xrightarrow{2} C \xrightarrow{1} A$$

5. (i) (a) $v - e + f = 2$

(b) If every face of $G$ is bounded by at least 4 edges, then

$$4f \leqslant 2e$$

[because, if every face is a 4-cycle, ea. edge of $G$ is on 2 faces + ea. face has 4 edges]

Substituting into Euler's formula:

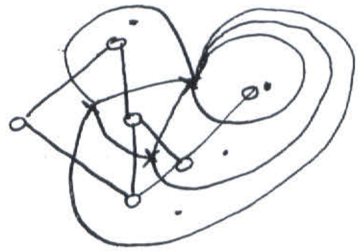
$$v - e + f = 2$$
$$4v - 4e + 4f = 8$$
$$4f = 8 - 4v + 4e \leq 2e$$
$$2e \leqslant 4v - 8$$
$$e \leq 2v - 4$$

(c) $K_{3,3}$ is bipartite, so it has no odd cycles (in part. no 3-cycles). Also $e = 9$, + $2v - 4 = 8$. So $e \nleq 2v - 4$. Hence $K_{3,3}$ is not planar.

5. (ii) (a) The graph $G$ is connected and has no bridges. Hence, by Robbins' Theorem, it is orientable.

(b) $G^*$ has 3 vertices and 6 faces

(c)



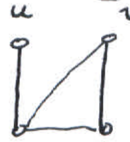
6. (i) The chromatic number for a graph is the minimum no. of colours needed to properly colour (the vertices of) $G$.

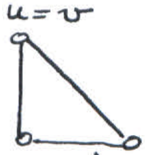
$P_G(t)$ is the no. of different colourings of a labelled simple graph $G$ from $t$ colours.

(ii) Let $G$ be a simple connected graph with largest-degree of $G$, $\Delta$ $(\geqslant 3)$.
If $G$ is not a complete graph then $G$ is $\Delta$-colourable.

$e = uv$

(iii)

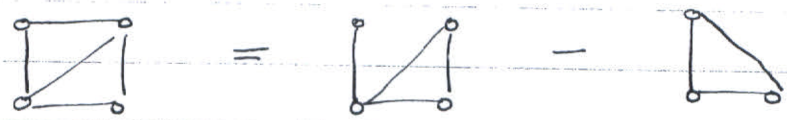



$P_G(t) = P_{G-e}(t) - P_{G\backslash e}(t)$ $e = uv$ exists in $G$

$P_G(t) = P_{G+e}(t) + P_{G\backslash e}(t)$ $u, v$ non-adj in $G$

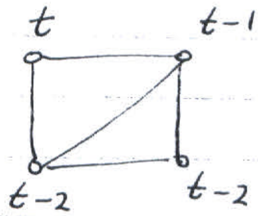
6. (iii) (cont'd)



G G-e P\e

$$P_G(t) = (t-1)\,t(t-1)(t-2) - t(t-1)(t-2)$$
$$= t(t-1)(t-2)\left[t-2\right]$$
$$= t(t-1)(t-2)^2.$$

OR




$$P_G(t) = t(t-1)(t-2)^2.$$

7. (i) A tree is a connected acyclic graph.

(ii) Let $P_1 = x_0 x_1 \ldots x_\ell$
$P_2 = x_0 y_1 \ldots y_k x_\ell$
be 2 distinct paths in a tree $T$.

Let $i+1$ be the minimal index s.t. $x_{i+1} \neq y_{i+1}$.
Let $j$ be the minimal index for which $j \geq i$ and $y_{j+1}$ is a vertex of $P_1$ (say $y_{j+1} = x_h$).
Then $x_i x_{i+1} \ldots x_h y_j y_{j-1} \ldots y_{j+1}$ is a cycle in $T$. A contradiction.
Thus, in $T$, any 2 distinct vertices are connected by a unique path.

(iii) No. of edges $= v-1$.

(iv)
$$\sum_{v \in V(T)} \deg(v) = 2 \times (\text{no. of edges}) = 2(v-1).$$

(v)
$$\sum_{v \in V(T)} \deg(v) = p + \sum_{\substack{\text{vertices} \\ \text{degrees} \geq 2}} \deg(v) \geq p + 2(v-p)$$
(there are $v-p$ of these)

$\therefore$ $p + 2(v-p) \geq 2(v-1)$.

or

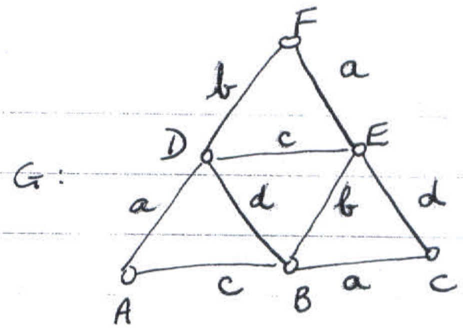
$p \geq 2$.

(vi) F is a forest, so each component is a tree. As there are 29 vertices and only 5 components, not all trees in F are isolated vertices. Hence, the chromatic no. of at least one component is 2 (the chromatic no. of a non-trivial tree is 2). Hence the chromatic no. of F is 2

8. (i) The minimum $k$ for which a graph is $k$-edge colourable is its edge chromatic no. (or its chromatic index).

(ii) If $G$ is a simple graph with largest degree $\Delta$, then $\chi'(G) = \Delta$ or $\Delta + 1$.

(iv)




Maximum degree $= 4$. So $\chi'(G) = 4$ or $5$.

A 4-edge colouring of $G$ is exhibited above.

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9. (i) A tournament $T$ is transitive if whenever $uv$ and $vw$ are arcs in $T$ then $uw$ is also an arc in $T$.

(ii) A sequence $s_1, s_2, \ldots, s_n$ of non-negative integers is called a score sequence of a tournament if $\exists$ a tournament (with $n$ vertices) whose vertices can be labelled $v_1, \ldots, v_n$ s.t. $\text{outdeg}(v_i) = s_i$ for $i \in \{1, \ldots, n\}$.

(iii) Let $T$ be a transitive tournament (with $n$ vertices).

Let $u$ and $w$ be 2 vertices of $T$.

We assume (without loss of generality) that $uw$ is an arc of $T$.

Let $W = \{v : wv \text{ is an arc of } T\}$

$=$ set of vertices adjacent from $w$

$\text{outdeg}(w) = |W|$.

For each $v \in W$, $wv$ is an arc of $T$.

As $T$ is transitive, $uv$ is an arc

[$uw$ is an arc, $wv$ is an arc, so $uv$ is an arc]

Thus, $\text{outdeg}(u) \geq |W| + 1$ and

hence $\text{outdeg}(u) \neq \text{outdeg}(w)$.

(iv) If there are 2 semi-Hamiltonian paths in a transitive tournament $T$, then there will be two vertices $x$ and $y$ s.t. in one path $x$ precedes $y$, and in the other, $y$ precedes $x$.

By transitivity $\exists$ an arc $xy$ and an arc $yx$ in $T$. This is a contradiction. Thus $\exists$ a unique semi-Hamiltonian path in $T$.