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EXAM.

1. (i) yes. distinct edges, distinct vertices.

(ii) no. ae is not an edge for example.

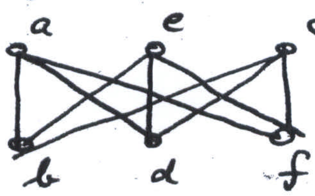
(iii) yes. degree of ea. vertex = 3.

(iv) (3, 3, 3, 3, 3, 3).

(v) $\sum_{v \in V} \deg(v) = 2e$; e = no. of edges

accept $\sum_{v \in V} \deg(v)$ is even.

(vi) Graph can be redrawn as

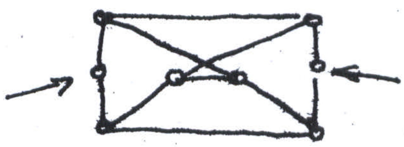


$\simeq K_{3,3}$ which is not planar.

(vii) No. All vertices are odd.

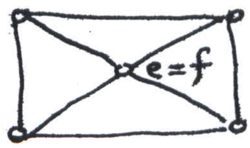
(viii) Hamiltonian cycle: e f c d a b e or application of Ore's or Dirac's Thms will do.

(ix)



extra vertices.

(x)



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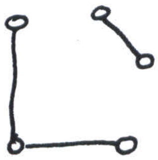
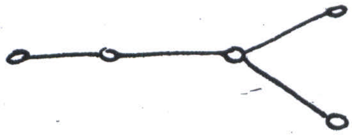
EXAM.

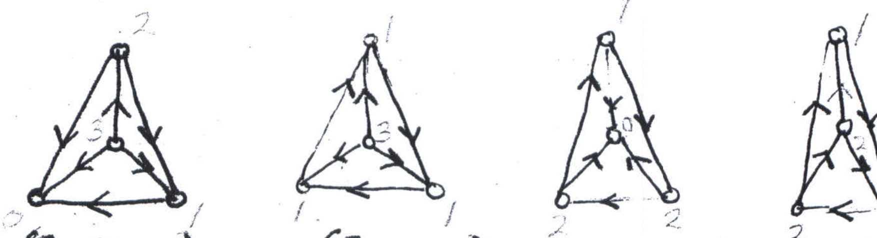
2. (i)



(ii)



(iii)

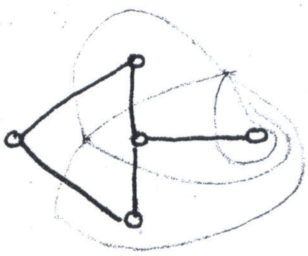


(3, 2, 1, 0) (3, 1, 1, 1) (2, 2, 1, 0) (2, 2, 1, 1)

(iv) A graph is planar iff it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.

K_5 is a subgraph of K_6. Hence by Kuratowski's Thm it is not planar.

(v)



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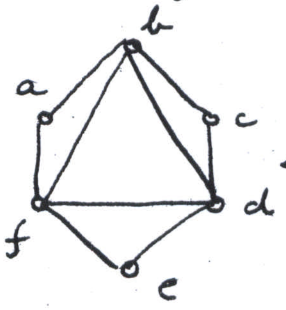
EXAM.

3. (i) yes. There exists a directed path between any two vertices. [label vertices clockwise around digraph as below:
e.g. a b c d e , c b d f a

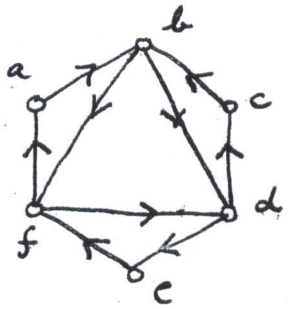
(ii)



(iii) yes. at most one arc from ea. vertex to another vertex.

(iv) yes. for ea. vertex : outdegree = indegree.

(v) no:



arcs form 3 disjoint directed cycles.

4. (i) Prüfer sequence $= (1,2,1,4,5,2)$.

T has 8 vertices

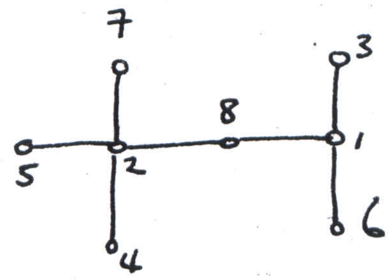
1	has degree	3
2	. . . . . .	3
3	. . . .	1
4	. . . .	2
5	. . . . .	2
6	. . . . .	1
7	. . . . .	1
8	. . . . .	1

degree sequence $= (1,1,1,1,2,2,3,3)$.

(ii) Prüfer sequence $= (1,2,2,1,2,8)$.

T has 8 vertices.

1	has degree	3
2		4
3	. . .	1
4		1
5		1
6		1
7		1
8		2



(iii) $19^{19-2} = 19^{17}$.

4.(iv)

K_n has n^{n-2} spanning trees each with $n-1$ edges.

Hence the total no. of all edges used in all spanning trees is $(n-1)n^{n-2}$.

Now each of the $\binom{n}{2}$ edges in K_n is equally likely to be included in a spanning tree.

Hence, the no. of spanning trees containing e is

$$\frac{(n-1)n^{n-2}}{\binom{n}{2}} = (n-1)n^{n-2} \times \frac{2}{n(n-1)}$$

$$= 2n^{n-3}.$$

5. (i)

$$A = \begin{array}{c c} \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \end{array} \qquad D = \begin{array}{c c} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \end{array}$$

$$M = D - A = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 3 & 0 & -1 & -1 \\ 0 & 0 & 2 & -1 & -1 \\ 0 & -1 & -1 & 2 & 0 \\ -1 & -1 & -1 & 0 & 3 \end{bmatrix}$$

$$\text{cofactor } (1,1) = (+1) \begin{vmatrix} 3 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 3 \end{vmatrix}$$

$$= (+1) \begin{vmatrix} 3 & 0 & -1 & -1 \\ -2 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 3 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} + 2 \begin{vmatrix} 0 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix}$$

$$= 3 \left\{ 2 \times 6 + 1(-3) + (-1)(2) \right\} + 2 \left\{ +1(-3) - 1(2) \right\}$$

$$= 3 \left\{ 12 - 3 - 2 \right\} + 2 \left\{ -5 \right\} = 3 \times 7 - 10 = 11$$

5. (ii) $K_{1,n} \cong K_{n,1}$ is a tree

Conversely, if a tree is complete bipartite, then it is $K_{m,n}$ for some m and n.

But $K_{m,n}$ has no vertices of degree 1 unless m or n is 1.

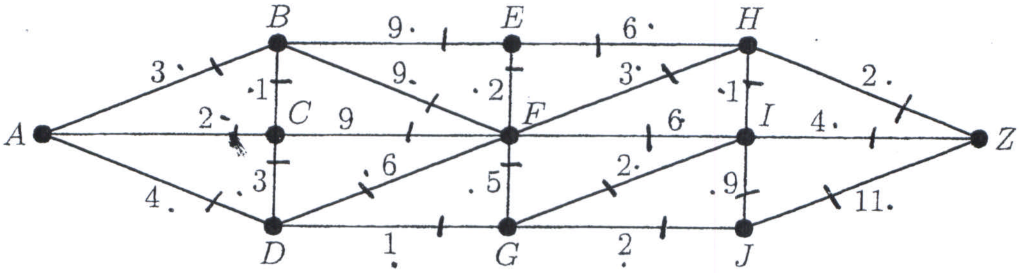
EXAM.

6. (i) ∘ List the edges in order of increasing weight

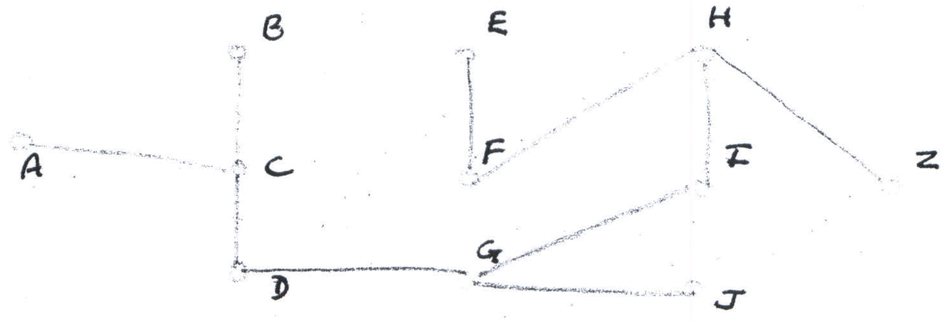
∘ Construct a tree by 1st drawing the vertices & no edges.

∘ Add an edge of least weight.

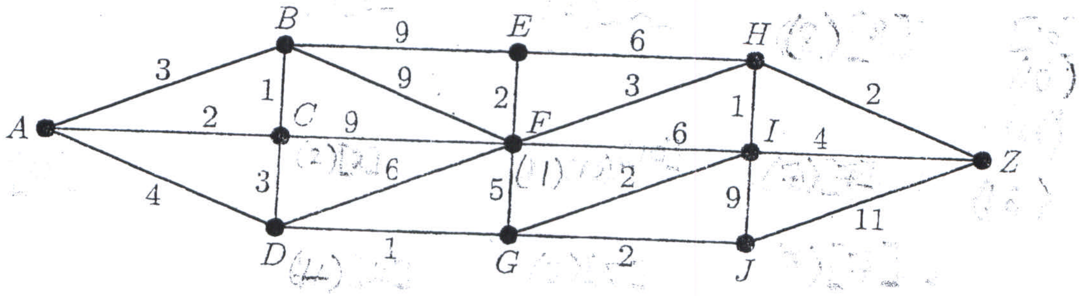
∘ Continue to add edges of least weight (without making a cycle), until we have a spanning tree.

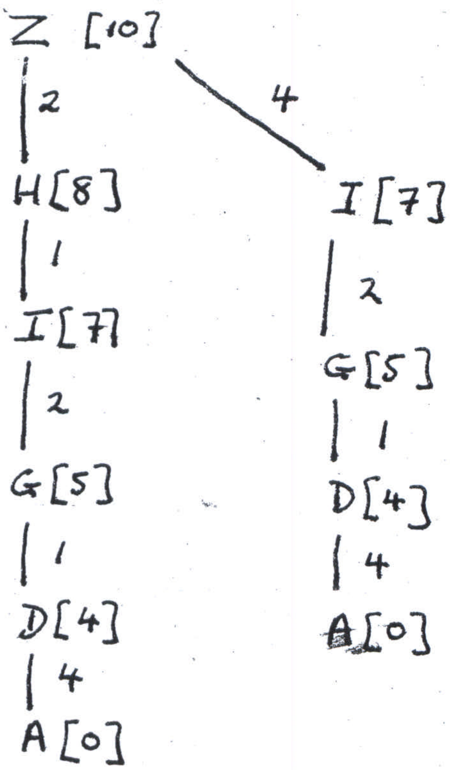


1	1	1	2	2	2	2	2	3	3	3	4	4	5	6	6	6	9	9	9	9	11
BC	DG	HI	AC	EF	GI	GJ	HZ	AB	CD	FH	AD	IZ	FG	DF	EH	FI	BE	BF	CF	IJ	JZ



6. (ii)


A
B
C
D
E
F
G
H
I
J
Z
3
2
4
1
3
9
9
9
6
1
2
5
1
2
6
3
6
1
9
2
2
4
11


Z [10]
2
4
H[8]
1
I[7]
2
G[5]
1
D[4]
4
A[0]
I[7]
2
G[5]
1
D[4]
4
A[0]

7. (i) $$v - e + f = 2.$$

(ii) Suppose that each face of G is a 4-cycle. Then ea. edge of G is on 2 faces + ea. face has 4 edges.

Thus $$4f = 2e.$$

Substituting into Euler's formula we get :

$$v - e + f = 2$$
$$4v - 4e + 4f = 8$$
$$\cancel{e =}$$
$$4v - 4e + 2e = 8$$
$$-2e = 8 - 4v$$
$$e = 2v - 4$$

If ea. face has at least 4 edges,
$$e \leq 2v - 4.$$

(iii) $K_{3,3}$ is bipartite so has no odd cycles (in part. no 3-cycles). So ea. face bounded by at least 4 edges.

Also, $e = 9$ + $2v - 4 = 8$ +
$$e \nleq 2v - 4$$
so $K_{3,3}$ is not planar.

8. (i) If G is a simple graph with largest vertex-degree Δ, then $\chi'(G) = \Delta$ or $\Delta + 1$.

(ii) If two edges have the same colour they cannot be incident with a common vertex.
If K_n, there are at most $\left(\frac{n-1}{2}\right)$ edges with the same colour, n odd.
Suppose we use $n-1$ colours, we can then colour at most $(n-1) \times \frac{n-1}{2}$ edges.
K_n has $\frac{n(n-1)}{2}$ edges.
Thus $(n-1)$ colours is not sufficient to colour K_n (n odd).
Vizing's Thm says that

$$\chi'(K_n) = n-1 \text{ or } n$$

Thus, if n is odd,

$$\chi'(K_n) = n.$$

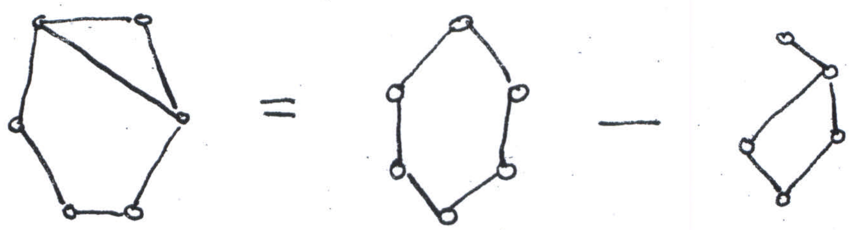
(iii) Since G is regular of degree 3, we have $\chi'(G) \geq 3$.
We obtain a 3-colouring of the edges of G by colouring the edges of a Hamiltonian cycle alternatively red + blue, then colour the remaining edges green.

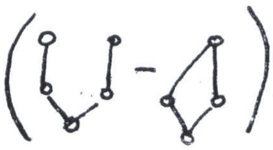
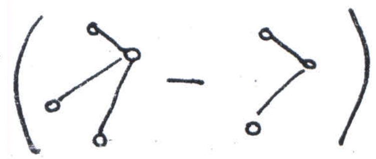
9. (i) $$P_G(t) = P_{G+e}(t) + P_{G/e}(t)$$

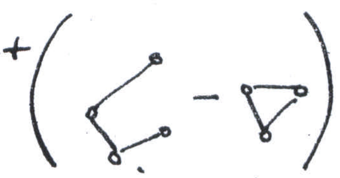
$e = uv$
u, v non-adjacent in G.

$$P_G(t) = P_{G-e}(t) - P_{G/e}(t)$$

$e = uv$
u, v adjacent in G.



$= T_6 -$  $- T_5 +$ 

$= T_6 - T_5 +$  $- T_5 + T_4 - T_3$

$$= T_6 - 2T_5 + 2T_4 - T_3 - K_3$$

$$= t(t-1)^5 - 2t(t-1)^4 + 2t(t-1)^3 - t(t-1)^2 - t(t-1)(t-2)$$

$$= t(t-1)\left[(t-1)^4 - 2(t-1)^3 + 2(t-1)^2 - (t-1) - (t-2)\right]$$

$$= t(t-1)\{t-2\}^2(t^2 - 2t + 2).$$

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EXAM.

9. (ii) $P_G(t) = t(t-1)(t-2)(t^2-5t+7)$

degree of $P_G(t) = 5$, so G has 5 vertices.

Expanding $P_G(t) = t^5 - 8t^4 + 24t^3 - 31t^2 + 14t$

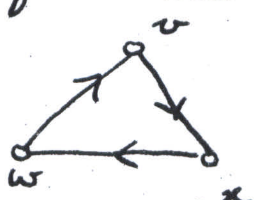
coeff. of $t^4 = -8$, so G has 8 edges.

10.

(i) The underlying graph for a tournament with n vertices $\cong K_n$. K_n has $\binom{n}{2}$ edges

Hence the tournament T has $\binom{n}{2}$ arcs.

(ii) Let w be any other vertex.
If v beats w, $\exists$ a path of length 1 from v to w and we are done.
Hence, suppose that w beats v: i.e. $w \to v$.
Among those vertices that v beats $\exists$ one say (x) which beats w [otherwise $s(w) \geqslant s(v)+1$ contradicting the max. score for v).
Hence, for some x, we have



,

& then $\exists$ a path of length 2 from v to w.

(iii) Suppose that T contains the cycle $v_1 v_2 \dots v_n v_1$. Since T contains no 3-cycle, $v_1 v_3$ must be an arc [consider vertices v_1, v_2, v_3].
Considering v_1, v_3, v_4, we see $v_1 v_4$ is an arc. Continuing, we have $v_1 v_n$ is an arc, contradicting $v_n v_1$ is an arc.
[In a tournament, exactly one of uv, vu is an arc, where u, v are any pair of vertices].