MATH 2969 EXAM JUNE, 2006

SOLUTIONS

1. (i) (a) yes
 (b) no
 (c) no

(ii) yes

(iii) no

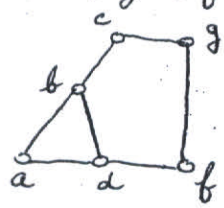
(iv) no

(v) yes

(vi) no

(vii) yes $a \to b \to c \to g \to f \to e \to d \to a$

(viii) $(3,3,2,2,2,2)$



(ix) 3

(x) degree sequence for G: $\begin{array}{ccccccc} e & b & d & c & g & f & a \\ (5, & 4, & 4, & 3, & 3, & 3, & 2) \end{array}$

$\sum$ degrees $= 5+4+4+3+3+3+2$

$= 24$

no. of edges in $G = 12$.

no. of edges in $K_7 = \frac{7\times 6}{2} = 21$.

so, no. of edges in $\overline{G} = 21 - 12 = 9$.

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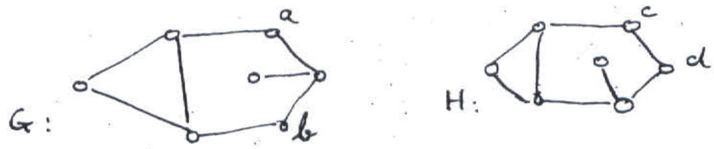
2. (i) The graphs are not isomorphic.



For the graph, G: the two vertices a, b of degree 2 are not adjacent; however, for graph, H: the two vertices of degree 2, c, d are adjacent.

(ii) Hand-shaking lemma:
In any graph, the sum of the degrees of the vertices is twice the no. of edges.

In G, suppose that there is an odd no. of vertices with odd degree.
Then $\sum_{v \in G}$ degrees $=$ an odd no.
$\neq 2 \times$ no. of edges
Hence there is a contradiction.
Thus, the no. of vertices of odd degree is even.

(iii) If G had an isolated vertex, the most no. of edges that G can have is

$$\binom{v-1}{2} = \frac{(v-1)(v-2)}{2}.$$

However, G has more than $\frac{(v-1)(v-2)}{2}$ edges.

Thus G has no isolated vertices.

(iv) Suppose that $\deg(v_0) \geqslant 3$ for some v_0.
Then $2n = \sum_{v \in V(G)} \deg(v) \geqslant 3 + 2(n-1) = 2n+1$,
which is a contradiction.
Hence, G is regular of degree 2.

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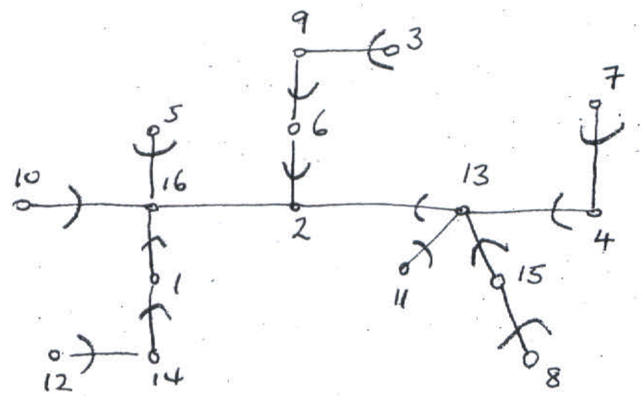
3. (i)

(a)



vertex	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
degree	2	3	1	2	1	2	1	1	2	1	1	1	4	2	2	4

(9, 16, 4, ~~15, 16, 13,~~ 13, 15, 6, 2, 16, 13, 14, 1, 16, 13, 2)

(b) (10, 10, 10, 3, 9, 8, 7, 1, 2).

We want a graph on 11 vertices. {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}
leaves are 4, 5, 6, 11

vertex	1	2	3	4	5	6	7	8	9	10	11
degree	2	2	2	1	1	1	2	2	2	4	1

(10, 10, 10, 3, 9, 8, 7, 1, 2)

{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}

edge (10, 4)

edge (10, 5)

edge (10, 6)

edge (3, 10)

edge (9, 3)

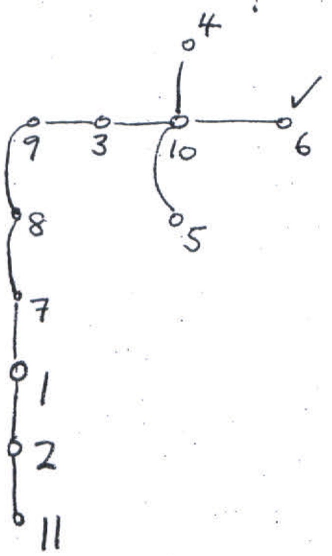
edge (8, 9)

edge (7, 8)

edge (1, 7)

edge (2, 1)

edge (2, 11)



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3. (ii) (a)

$$A = \begin{array}{c|cccc} & a & b & g & f \\ \hline a & 0 & 1 & 1 & 0 \\ b & 1 & 0 & 1 & 1 \\ g & 1 & 1 & 0 & 1 \\ f & 0 & 1 & 1 & 0 \end{array}$$

$$\text{trace of } A^3 = 6 \times \text{no. of triangles in } G$$
$$= 6 \times 2$$
$$= 12.$$

(b) Let G be a connected simple labelled graph with adjacency matrix A and degree matrix D.

Then all cofactors of $M = D - A$ are equal and their common value is the no. of spanning trees of G.

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$M = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

$$\text{cofactor} = (-1) \begin{vmatrix} -1 & -1 & 0 \\ 3 & -1 & -1 \\ -1 & 3 & -1 \end{vmatrix}$$

3.(ii) (b) (cont'd)

$$= (-1)\begin{vmatrix} -1 & -1 & 0 \\ 4 & -4 & 0 \\ -1 & 3 & -1 \end{vmatrix} \qquad R_2 = R_2 - R_3$$

$$= (-1)(-1)\begin{vmatrix} -1 & -1 \\ 4 & -4 \end{vmatrix}$$

$$= (-1)(-1)(4+4) = 8.$$

4. (i) Given a connected weighted graph, G, with n vertices.

(1) List the edges in order of increasing weight

(2) Construct the tree by first drawing the n vertices (with no edges)

(3) Add an edge of least weight.

(4) Continue adding edges of least weight, without making a cycle, until we have a spanning tree (i.e. have n-1 edges).

AB ✓ 8 ✓
AC ✓ 1
BD ✓ 4 ✓
BC ✓ 7 ✓
CD 2 ✓
CE 13 ✓
DE 10 ✓
DF 12 ✓
EF 2 ✓
EG 1 ✓
FG 1 ✓
FH 1
GH 3 ✓
BE 5 ✓

The algorithm is adapted by:

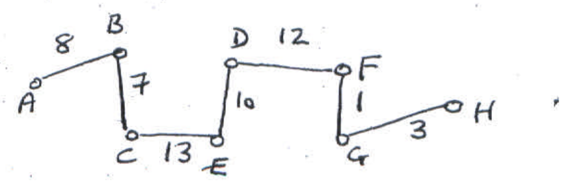
(a) starting with GF (whatever its weight is)

(b) listing the edges in decreasing weight order

(c) adding edges of most weight.

✓	✓	✓	✓	✓	✓			✓				
GF	CE	DF	DE	AB	BC	BE	BD	GH	CD	EF	EG	FH
1	13	12	10	8	7	5	4	3	2	2	1	1

Eight vertices, so need $8-1=7$ edges



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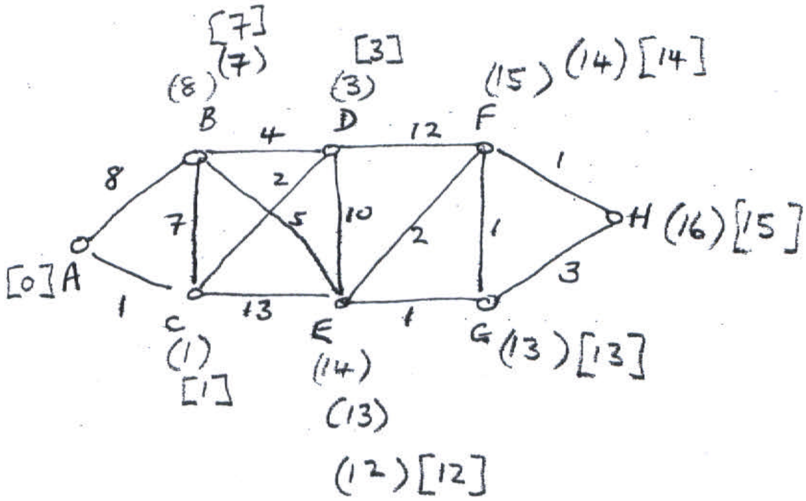
4. (ii)



$H \xrightarrow{1} F \xrightarrow{1} G \xrightarrow{1} E \xrightarrow{5} B \xrightarrow{4} D \xrightarrow{2} C \xrightarrow{1} A$

$F \xrightarrow{2} E \xrightarrow{5} B \xrightarrow{4} D \xrightarrow{2} C \xrightarrow{1} A$

5. (i) The chromatic number, $\chi(G)$, for a graph G, is the minimum no. of colours needed to properly colour G.

(ii) The proof is by induction on the no. of vertices of G.

Let G be a simple graph with n vertices.

Let v be any vertex of G.

$H = G - v$ is a simple graph with $n-1$ vertices and largest vertex degree at most Δ.

By induction hypothesis, H is $(\Delta+1)$-colourable.

A $(\Delta+1)$-colouring for G is then obtained by colouring v with a different colour from the (at most Δ) vertices adjacent to v.

$\chi(G)$ is the minimum no. of colours needed to colour G. Hence $\chi(G) \leq \Delta+1$.

5. (iii) We use induction on e, the no. of edges of G.
When $e=0$, $G=\overline{K}_n$ and $P_G(t)=t^v$.
Let G be a graph with v vertices with $e\geq 1$.
Now $G-f$ and $G\backslash f$, where f is an edge in G, each have fewer edges than G. $G\backslash f$ has $v-1$ vertices.

By the induction hypothesis, $\exists$ non-negative integers $\{a_i\}$ and $\{b_i\}$ s.t.

$v-(v-1)$

$$P_{G-f}(t)=\sum_{i=0}^{v-1}(-1)^i a_i t^{v-i} \quad \text{and}$$

$$P_{G\backslash f}(t)=\sum_{i=0}^{v-2}(-1)^i b_i t^{v-i}$$

That is, $P_{G-f}(t)=t^v-(e-1)t^{v-1}+a_2t^{v-2}-\dots$

$P_{G\backslash f}(t)=t^{v-1}-b_1t^{v-2}+b_2t^{v-3}-\dots$

$$P_G(t)=P_{G-f}(t)-P_{G\backslash f}(t)$$

Thus $P_G(t)=t^v-et^{v-1}+(a_2+b_1)t^{v-2}-\dots$.

Thus $P_G(t)$ is a polynomial of degree v with leading coeff. $a_0=1$ and next coefficient is $-e$

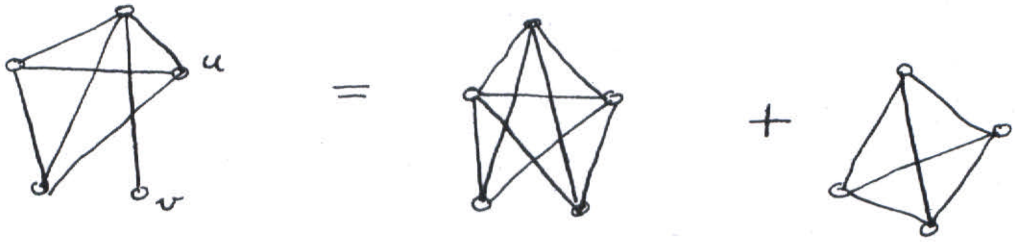
(iv) $P_G(t)=P_{G-e}(t)-P_{G\backslash e}(t)$ $\quad$ $e=uv$ is an edge in G

$P_G(t)=P_{G+e}(t)+P_{G\backslash e}(t)$ $\quad$ $e=uv$ where u,v are not adjacent in G.

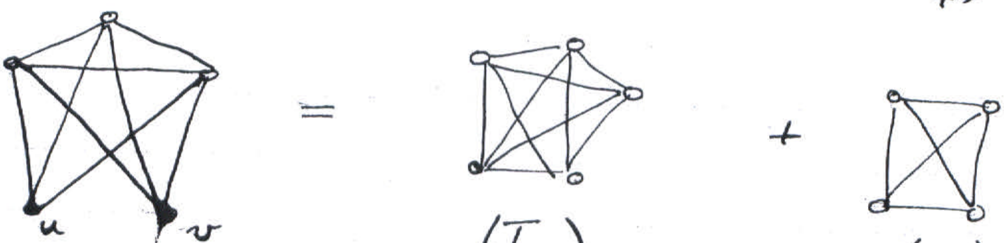
5. (iv) (cont'd)

We call the 5 vertex component of H, G. The 3-vertex component is called F.

$$P_F(t) = t(t-1)(t-2) \qquad (F \simeq T_3.)$$



(T_4)



(T_5) (T_4)

Thus

$$P_G(t) = P_{T_5}(t) + 2\,P_{T_4}(t)$$

$$= t(t-1)(t-2)(t-3)(t-4) + 2t(t-1)(t-2)(t-3)$$

$$= t^5 - 8t^4 + 23t^3 - 28t^2 + 12t$$

Thus $P_H(t) = P_G(t)\ P_F(t)$

$$= (t^5 - 8t^4 + 23t^3 - 28t^2 + 12t)(t)(t-1)(t-2)$$

$$= t^8 - 11t^7 + 49t^6 - 113t^5 + 142t^4 - 92t^3 + 24t^2$$

6. (i) (a) $$v - e + f = 2$$

(ii) (b) Suppose that every face is an n-cycle. Then each edge of G on 2 faces & each face has n edges.

Thus $$nf = 2e.$$

Substituting into Euler's formula, we get

$$v - e + f = 2$$

$$nv - ne + nf = 2n$$

so $$e = \frac{n(v-2)}{n-2}$$

Thus, if each face is bounded by at least n edges, we have

$$e \leq \frac{n(v-2)}{n-2}$$

(c) Thus, by part (b), if ~~any~~ G is any simple planar graph with $v \geqslant 3$ vertices and e edges

$$e \leq 3v - 6$$

If G has no triangles, then

$$e \leq 2v - 4.$$

For K_5: $v = 5$, $e = 10$, but

$$e = 10 > 9 = 3v - 6,$$

so K_5 cannot be planar.

For $K_{3,3}$: this graph is bipartite and hence has no triangles.

$$e = 9 \quad \text{and} \quad 2v - 4 = 8$$

so $e \nleq 2v - 4$.

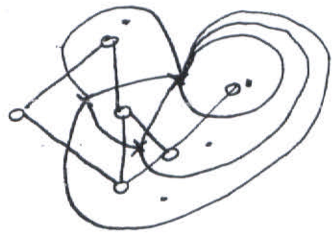
Thus $K_{3,3}$ cannot be planar.

6. (i) (d) If the vertices of degree 1 are removed together with the edges with which they are incident, we have a subgraph H which has $30-15 = 15$ vertices. H is connected and planar. Hence the no. of edges of H is $\leq 3(15)-6 = 39$. As 15 edges were removed from G, the no. of edges in $G \leq 39+15 = 54$.

(ii) (a) The graph G is connected and has no bridges. Hence, by Robbins' Theorem, it is orientable.

(b) G^* has 3 vertices and 6 faces

(c)



7. (i) A semi-Eulerian trail is a trail containing each vertex and each edge, but is not a closed trail.

A tree is a connected acyclic graph.

(ii) Let $P_1 = x_0 x_1 \dots x_l$
$P_2 = x_0 y_1 \dots y_k x_l$
be 2 distinct paths in a tree T.

Let $i+1$ be the minimal index s.t. $x_{i+1} \neq y_{i+1}$.
Let j be the minimal index for which $j \geq i$ and y_{j+1} is a vertex of P_1 (say $y_{j+1} = x_h$).
Then $x_i x_{i+1} \dots x_h y_j y_{j-1} \dots y_{j+1}$ is a cycle in T. A contradiction.
Thus, in T, any 2 distinct vertices are connected by a unique path.

(iii)
$$\sum_{v \in V(T)} \deg(v) = 2 \times (\text{no. of edges}) = 2(v-1).$$

$$\sum_{v \in V(T)} \deg(v) = p + \sum_{\substack{\text{vertices} \\ \text{degrees} \geq 2 \\ (\text{there are } v-p \text{ of these})}} \deg(v) \geq p + 2(v-p)$$

$$\therefore \quad p + 2(v-p) \geq 2(v-1).$$

or

$$p \geq 2.$$

Consider a tree T with v vertices.

7. (iv) Suppose that T has a semi-Eulerian trail. Then T must have exactly vertices of odd degree. By part (iii), T has at least two vertices of degree 1. Thus T has exactly two vertices of degree 1.

Suppose that T has exactly 2 vertices of degree 1. Now T has $v-1$ edges, so the sum of the degrees of $T = 2(v-1) = 2v-2$. Thus the remaining $v-2$ vertices have degrees totalling $2v-4 = 2(v-2)$. However, each of these vertices has degree ≥ 2. Thus, each of these vertices has degree exactly 2. Hence T has exactly two odd vertices & so is semi-Eulerian.

(v) F is a forest, so each component is a tree. As there are 29 vertices and only 5 components, no all trees in F are isolated vertices. Hence, the chromatic no. of at least one component is 2 (the chromatic no. of a non-trivial tree is 2). Hence the chromatic no. of F is 2.

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8. (i) The minimum k for which a graph is k-edge colourable is its edge chromatic no. (or its chromatic index).

(ii) If G is a simple graph with largest degree Δ, then $\chi'(G) = \Delta$ or $\Delta+1$.

(iii) Firstly, we observe that if 2 edges have the same colour they cannot be incident with the same vertex.
So, in K_n, (n odd) we can colour at most $\frac{n-1}{2}$ edges with the same colour.
In K_n (n odd), using $n-1$ colours we can colour at most $(n-1) \times \frac{n-1}{2}$ edges.
But K_n has $\frac{n(n-1)}{2}$ edges. Thus $n-1$ colours is insufficient to colour K_n (n odd).
Vizing's Theorem says $\chi'(K_n) = n-1$ or or n.

Thus $\chi'(K_n) = n$ (n odd).

To colour the edges of K_n (n even) with $n-1$ colours, we proceed as follows:

1/ Colour the subgraph K_{n-1} ($n-1$ is odd) with the colours $i \in \{1, \dots, n-2\}$.
N.B. Each vertex in K_{n-1} is missing 1 colour.

2/ Label these vertices v_i, $i \in \{1, \dots, n-1\}$ of K_{n-1} s.t. v_i is the vertex missing colour i.

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8. (iii) 3/ Add the extra vertex v & the edges vv_i, $i \in \{1, \dots, n-1\}$ to form K_n.

4/ Colour the edges vv_i, $i \in \{1, \dots, n-1\}$, with the colour i.

K_n, n even, is thus coloured with $n-1$ colours.

Vizing's Thm says that $\chi'(K_n) = n$ or $n-1$ (n even)

We have shown that $\chi'(K_n) = n-1$ (n even).

(iv) Since H is regular of degree 3,
$$\chi'(H) = 3 \text{ or } 4.$$
To obtain a 3-colouring of the edges of H, we colour the Hamiltonian cycle alternately black and white, then colour the remaining edges blue.

9. (i) A tournament T is transitive if whenever uv and vw are arcs in T then uw is also an arc in T.

(ii) A sequence $s_1, s_2, \ldots, s_n$ of non-negative integers is called a score sequence of a tournament if $\exists$ a tournament (with n vertices) whose vertices can be labelled $v_1, \ldots, v_n$ s.t. $\text{outdeg}(v_i) = s_i$ for $i \in \{1, \ldots, n\}$.

(iii) Let T be a transitive tournament (with n vertices).

Let u and w be 2 vertices of T.

We assume (without loss of generality) that uw is an arc of T.

Let $W = \{v : wv \text{ is an arc of } T\}$
$= \text{set of vertices adjacent from } w$

$\text{outdeg}(w) = |W|$.

For each $v \in W$, wv is an arc of T.

As T is transitive, uv is an arc

[uw is an arc, wv is an arc, so uv is an arc]

Thus, $\text{outdeg}(u) \geq |W| + 1$ and hence $\text{outdeg}(u) \neq \text{outdeg}(w)$.

(iv) If there are 2 semi-Hamiltonian paths in a transitive tournament T, then there will be two vertices x and y s.t. in one path x precedes y, and in the other, y precedes x.

By transitivity $\exists$ an arc xy and an arc yx in T. This is a contradiction. Thus $\exists$ a unique semi-Hamiltonian path in T.