THE UNIVERSITY OF SYDNEY

MATH2969/2069

Graph Theory	**Tutorial 1 (Week 8)**	**2008**

1. Draw a picture of each of the following graphs, and state whether or not it is simple.

(a) $G_1 = (V_1, E_1)$, where $V_1 = \{a, b, c, d, e\}$ and
$E_1 = \{ab,\ bc,\ ac,\ ad,\ de\}$.

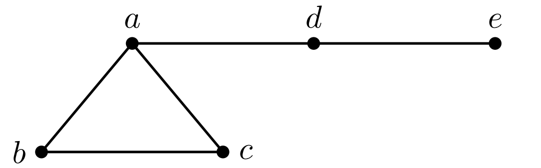
(b) $G_2 = (V_2, E_2)$, where $V_2 = \{P, Q, R, S, T\}$ and
$E_2 = \{PQ,\ PR,\ PS,\ PT,\ TR,\ PR\}$.

(c) $G_3 = (V_3, E_3)$, where $V_3 = \{v_1, v_2, v_3, v_4, v_5\}$ and
$E_3 = \{v_1v_1,\ v_1v_2,\ v_2v_3,\ v_3v_4,\ v_5v_4,\ v_4v_5\}$.

Solution.

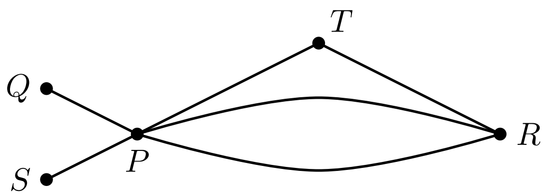
A *graph* consists of a non-empty finite set of *vertices*, and a finite set of *edges*, each edge corresponding to a pair of vertices. The graph is *simple* if (i) no two edges correspond to the same pair of vertices (that is, there are no multiple edges), and (ii) each edge corresponds to a pair of *distinct* vertices (that is, there are no loops). A non-simple graph is sometimes called a *pseudograph*.

(a)



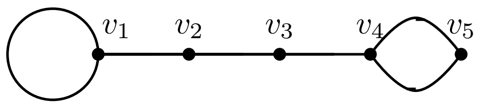
Simple.

(b)



Not simple.

(c)



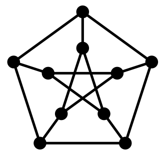
Not simple.

Note that there are two edges connecting P and R in (b), and so (b) is not simple.

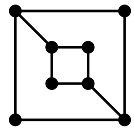
Since (c) has a loop, (c) is not simple.

2. For each of the following graphs write down the number of vertices, the number of edges and the degree sequence. Verify the hand-shaking lemma in each case.

(*i*)



(*ii*)



Solution.

The *degree* of a vertex is the number of edges incident at that vertex. (Note that a loop on a vertex contributes 2 to the degree of that vertex.)

The *degree sequence* of a graph is a list of the degrees of the vertices, in ascending order.

The hand-shaking lemma: Sum of degrees $= 2\times$ number of edges.

Corollary: The number of vertices of odd degree is even.

(*i*) The graph has 10 vertices, 15 edges and degree sequence $(3, 3, 3, 3, 3, 3, 3, 3, 3, 3)$.

The hand-shaking lemma is verified: Sum of degrees $= 30 = 2 \times 15$.

(*ii*) The graph has 8 vertices, 10 edges and degree sequence $(2, 2, 2, 2, 3, 3, 3, 3)$.

The hand-shaking lemma is verified: Sum of degrees $= 20 = 2 \times 10$.

3. A sequence $d = (d_1, d_2, \ldots, d_n)$ is *graphic* if there is a simple graph with degree sequence d. Determine whether or not the following sequences are graphic. If the sequence is graphic, draw a corresponding graph.

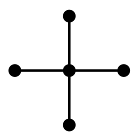
(a) $(2, 3, 3, 4, 4, 5)$ (b) $(2, 3, 4, 4, 5)$ (c) $(1, 1, 1, 1, 4)$
(d) $(1, 3, 3, 3)$ (e) $(1, 2, 2, 3, 4, 4)$ (f) $(1, 3, 3, 4, 5, 6, 6)$
(g) $(2, 2, 2)$

Solution.

(a) The sum of the terms, $2 + 3 + 3 + 4 + 4 + 5$, is odd. Equivalently, the number of odd terms, 3, is odd. The corollary to the hand-shaking lemma states that for any graph the number of vertices of odd degree must be even, so no graph can have this degree sequence.

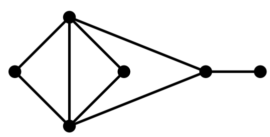
(b) A simple graph with n vertices cannot have a vertex of degree more than $n - 1$. Here $n = 5$.

(c) This sequence is graphic:



(d) Suppose a graph exists with such a degree sequence. Then each vertex of degree 3 has an edge leading to each of the other vertices, so the fourth vertex must have degree at least 3 — a contradiction.

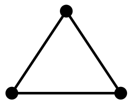
(e) This sequence is graphic:



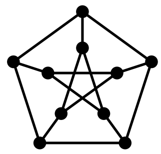
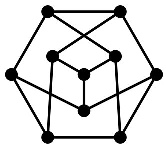
(f) Not possible. If a simple graph has 7 vertices, then the maximum degree of any vertex is 6, and if two vertices have degree 6 then all other vertices must have degree at least 2.

Alternately:
Suppose a graph exists with such a degree sequence. Removing the vertex of degree 1 and its incident edge leaves a graph with 6 vertices and at least one vertex of degree 6 — impossible (see (b) with $n = 6$).

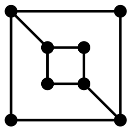
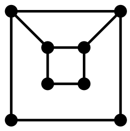
(g) This sequence is graphic:



4. (a) By suitably lettering the vertices, prove that the following two graphs are isomorphic:

G :  G' : 

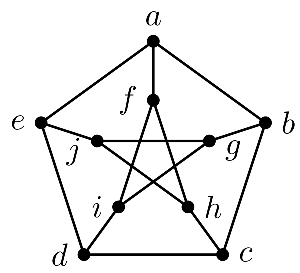
(b) Explain why (i.e., prove that) the following two graphs are not isomorphic:

H :  H' : 

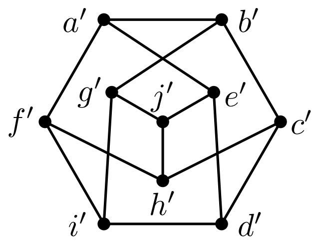
Solution.

Two graphs G_1 and G_2 are *isomorphic* if there is a bijection f between the vertices of G_1 and G_2 such that the number of edges joining any two vertices v and w in G_1 is equal to the number of edges joining $f(v)$ and $f(w)$ in G_2. (In other words, f is a bijective function between the vertices of G_1 and G_2 which preserves adjacencies.)

(a) In G label the vertices a to j in any order, as in the diagram, say. We wish to label the vertices of G' by a' to j', so that a corresponding to a', ..., j corresponding to j' is an isomorphism. All vertices have the same degree, 3, so maybe it doesn't matter which vertex of G' is a', but then we must be careful to label the vertices adjacent to a' as b', e', f' (since a is adjacent to b, e, f), but maybe the order of b', e', f' won't matter.



Proceeding in this way, with suitable care, we arrive at something like the following labelling of G': Now we have to check that all adjacencies are preserved by this correspondence, i.e., for each x, y in G, x is adjacent to y if and only if x' is adjacent to y' in G'. We can do this by listing all the edges. In G the edges are:
$ae, af, ab, bg, bc, ch, cd, di, de, ej, fi, fh, gi, gj, hj$.
In G' the edges are:
$a'e', a'f', a'b', b'g', b'c', c'h', c'd', d'i', d'e', e'j', f'i', f'h', g'i', g'j', h'j'$. It is clear from these lists that the adjacencies are preserved, and hence the correspondence is an *isomorphism*, and G and G' are *isomorphic*.



(b) H and H' have the same number of vertices and the same number of edges, and they even have the same degree sequence $(2,2,2,2,3,3,3,3)$, which means they may be isomorphic (but only maybe).

In fact, they are not, because in H no two vertices of degree 2 are adjacent, whereas in H' pairs of vertices of degree 2 are adjacent. This could not happen if there was an isomorphism between H and H' (i.e., a one to one correspondence between the vertices of H and the vertices of H' which preserves adjacencies).

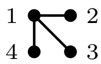
The moral is, that while certain conditions are certainly necessary for graphs to be isomorphic (same numbers of vertices, same numbers of edges, same degree sequences, to mention just three), they are not sufficient.

The only way to prove two graphs are isomorphic is to find an isomorphism. (This is exactly what we did in (a).) To show graphs are not isomorphic, we need only find just one condition, known to be necessary for isomorphic graphs, which does not hold.

5. List all non-identical simple labelled graphs with 4 vertices and 3 edges. How many of these are not isomorphic as unlabelled graphs?

Solution.

With 4 vertices (labelled 1,2,3,4), there are $\binom{4}{2} = 6$ possible edges: $[1,2]$, $[1,3]$, $[1,4]$, $[2,3]$, $[2,4]$, $[3,4]$. From these six we choose 3 to make a labelled graph with 4 vertices and 3 edges: $\binom{6}{3} = 20$ ways to choose the three edges.

For example, choosing $[1,2]$, $[1,3]$, $[1,4]$ gives 

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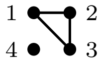
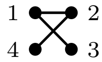
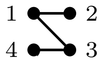
$[1,2]$, $[1,3]$, $[2,3]$ gives 

; $[1,2]$, $[1,3]$, $[2,4]$ gives 

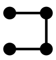
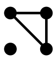
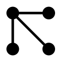
;

$[1,2]$, $[1,3]$, $[3,4]$ gives 

etc.

If we disregard labels, the fourth of these is isomorphic to the third, each being isomorphic to (the linear or line graph with four vertices).

In fact, among the twenty distinct labelled graphs there are only three non-isomorphic as unlabelled graphs:

 (12 of the 20),  (4 of the 20),  (4 of the 20).

6. (i) What is the maximum number of edges in a simple graph on n vertices?

(ii) How many simple labelled graphs with n vertices are there?

Solution.

(i) The maximum number of edges exists when each pair of vertices is joined. Since there are $\binom{n}{2}$ pairs of vertices, the maximum number of edges is $\binom{n}{2} = \dfrac{n(n-1)}{2}$.

Another way to think about this question is as follows. Each vertex can be joined to at most $n-1$ other vertices. Therefore, the sum of the degrees is at most $n \times (n-1)$, and the maximum number of edges is $n \times (n-1)/2$.

(*ii*) There is one labelled graph with 0 edges. There are $\binom{\binom{n}{2}}{1}$ different labelled graphs with 1 edge. For each $k = 0, 1, 2, \ldots, n$ there are $\binom{\binom{n}{2}}{k}$ different labelled graphs. The number of simple labelled graphs with n vertices is therefore

$$\binom{\binom{n}{2}}{0} + \binom{\binom{n}{2}}{1} + \binom{\binom{n}{2}}{2} + \ldots + \binom{\binom{n}{2}}{n} = (1+1)^{\binom{n}{2}} = 2^{\binom{n}{2}}.$$

Another way to think about this: Each of the $\binom{n}{2}$ edges can either be included in the graph or not. That is, there are 2 possibilities for each edge, and hence $2^{\binom{n}{2}}$ different simple labelled graphs.

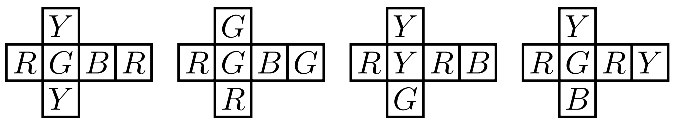
7. Explain why, in the solution using Graph Theory of the 'Instant Insanity' puzzle, the chosen subgraphs should satisfy the conditions that they

(a) have no edges in common;

(b) contain exactly one edge from each cube;

(c) contain only vertices of degree two.

Illustrate your answer by finding a solution for the following set of cubes.



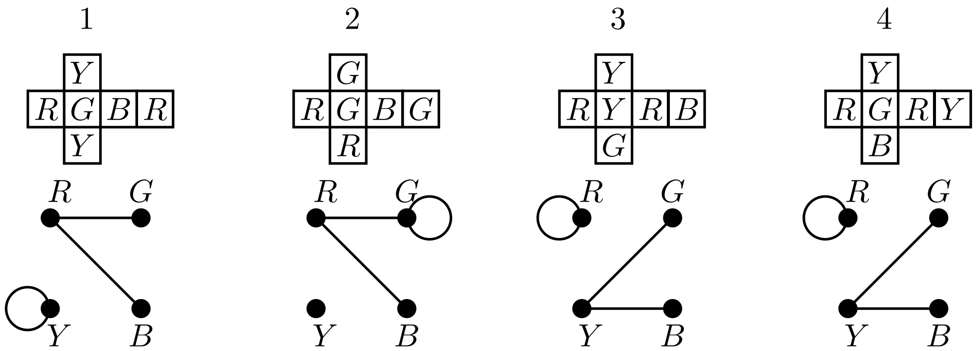
(There are several solutions.)

Solution.

To solve the 'Instant Insanity' puzzle, we need to stack four given cubes, each face coloured either R or G or B or Y, into a column so that each of the four sides of the column has each of the four colours occurring exactly once.

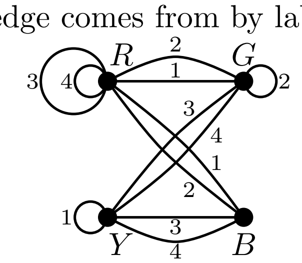
(To interpret the above four figures, imagine cutting each of them out along the outside edges, then folding them along the other edges into cubes.)

We first construct a graph for each cube, the vertices being the four possible colours, and with an edge joining two colours X, Y whenever X and Y appear on opposite faces.



We combine these four graphs into one graph G, keeping track of which cube each edge comes from by labelling the edges:

G =



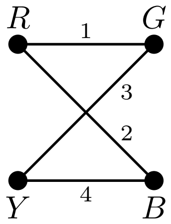
We need to find subgraphs H_{LR} and H_{FB} of G, where H_{LR} determines the Left-Right colours of our column of cubes and H_{FB} determines the Front-Back colours. To give a solution, H_{LR} and H_{FB} must

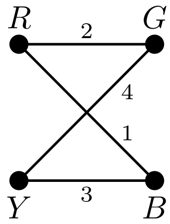
(a) have no edges in common, because opposite faces of a cube cannot be both L-R and F-B at the same time;

(b) contain exactly one edge from each cube, because each cube is in the column;

(c) contain only vertices of degree 2, because each face of the column has each colour occurring exactly once.

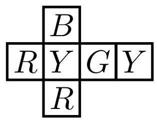
It is not difficult to find suitable subgraphs H_{LR} and H_{FB}. One solution is

$H_{LR} =$ 

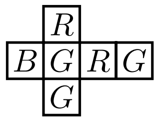
$H_{FB} =$ 

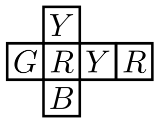
The edges marked 1 in H_{LR} and H_{FB} tell us to orientate Cube 1 so that RG is left-right (or right-left), and RB is front-back (or back-front) — 4 possible orientations. The other edges then determine the orientation for each of Cubes 2, 3 and 4. Thus from H_{LR} Cube 2 must have BR as left-right (not right-left since Cube 1 already has R left) and from H_{FB} Cube 2 must have GR as front-back (not back-front with Cube 1 already R front).

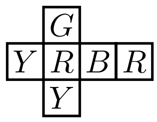
Continuing thus, we are able to stack the cubes in the following column:

Cube 1 

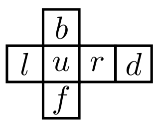
(on top)

Cube 2 

Cube 3 

Cube 4 

(on the bottom)

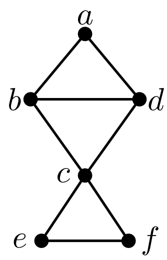
where the orientation of each cube is given by 

(l = left, r = right, f = front, b = back, u = upper side, d = down side).

We can check that each colour occurs exactly once in each of these positions. For example, Cubes 1, 2, 3, 4 have on the left colours R, B, G, Y respectively. (We are, of course, concerned with the colours in the l, r, f and b positions only. The colours in the u or d positions do not matter. Also the order of the cubes in the stack does not matter — Cube 1 could be top, bottom, or anywhere in between.)

Can you find any alternative choice for the subgraphs H_{LR}, H_{FB} to give an alternative solution?

8. For each of the following sequences of vertices, state whether or not it represents a walk, trail, path, closed walk, closed trail, or cycle in the graph illustrated.



(i) $abcefcbd$ (ii) $abcefcd$

(iii) $abcefcdba$ (iv) $bcefcdb$

(v) $bcdb$ (vi) $abefcd$

Solution.

A *walk* is a finite sequence of edges of the form $v_1v_2, v_2v_3, v_3v_4, \ldots, v_{n-1}v_n$ in which any two consecutive edges are adjacent or identical. If $v_1 = v_n$ the walk is *closed.*

A *trail* is a walk in which all the edges are distinct.

A *path* is a trail in which all the vertices (except possibly v_1 and v_n) are distinct. A closed path (with at least one edge) is a *cycle*.

(i) A walk. (ii) A trail (and also a walk). (iii) A closed walk.

(iv) A closed trail (and also a closed walk).

(v) A cycle (and a closed trail, and a closed walk).

(vi) This is not a walk of any type.

9. Suppose that a graph G is regular of degree r, where r is odd.

(i) Prove that G has an even number of vertices.

(ii) Prove that the number of edges in G is a multiple of r.

Solution.

(i) Let n be the number of vertices. Then, since each vertex has degree r, the sum of the degrees is $n \times r$, and this must be an even number (by the hand-shaking lemma). Now, r is odd, and so n must be even.

(ii) The number of edges is $\dfrac{n \times r}{2}$. (The hand-shaking lemma again.)

Since n is even, $n = 2k$ for some integer k, and hence the number of edges is $k \times r$.

10. Let u and v be distinct vertices of a graph. Prove that there is a walk from u to v if and only if there is a path from u to v.

Solution.

Since a path *is* a walk (without repeated vertices), if there is a path from u to v then there is a walk from u to v.

Now suppose that there is a walk W from u to v which is not a path – that is, a walk containing repeated vertices. Between any two occurrences of a repeated vertex in W, there is a closed walk. Remove from W all such closed walks, and the remaining edges will form a path from u to v.

11. A simple graph has 20 vertices. Any two distinct vertices u and v are such that deg $(u)+$ deg$(v) \geq 19$. Prove that the graph is connected.

Solution.

Suppose the graph is disconnected, with $k \geq 1$ vertices in one component, and $20 - k$ vertices in another. Then the maximum possible degree for a vertex in the first component is $k - 1$, and in the second is $19 - k$. Therefore, if a vertex u is in the first component, and a vertex v is in the other component, $\deg(u) + \deg(v) \leq (k-1) + (19-k) = 18$. But $\deg(u) + \deg(v) \geq 19$ for all distinct vertices u and v. We have a contradiction, and so the graph must be connected.

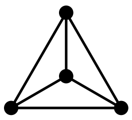
12. (*i*) Draw the graphs formed by the vertices and edges of a tetrahedron, a cube, and an octahedron.

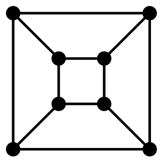
(*ii*) Are any of these graphs Eulerian?

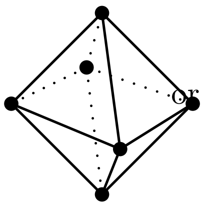
(*iii*) Find a Hamiltonian cycle in each graph.

Solution.

(*i*) Tetrahedron: Cube: Octahedron:







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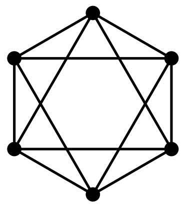

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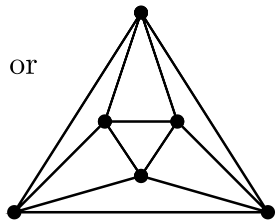


(*ii*) A connected graph is *Eulerian* if there is a closed trail which uses every edge exactly once.

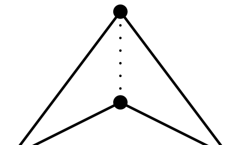
A connected graph is *semi-Eulerian* if it is not Eulerian, but there is a trail which uses every edge exactly once.

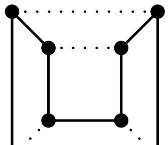
A connected graph is Eulerian if and only if the degree of each vertex is even.

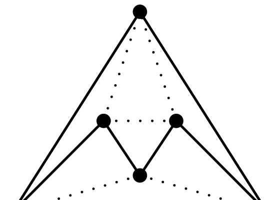
So the graph of the octahedron is the only one which is Eulerian.

(*iii*) A graph is *Hamiltonian* if there exists a closed path which passes through every vertex of the graph exactly once. Such a path is called a *Hamiltonian* cycle.

The solid lines in the graphs below are Hamiltonian cycles.







13. (a) Is it possible to draw a sketch of K_5 without lifting your pen from the paper, and without retracing any edges?

(b) For which values of n (≥ 2) is K_n (i) Eulerian? (ii) semi-Eulerian?

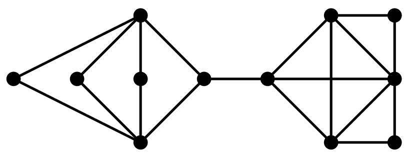
Solution.

(a) The degree of each vertex in K_5 is 4, and so K_5 is Eulerian. Therefore it can be sketched without lifting your pen from the paper, and without retracing any edges.

(b) (i) In K_n the degree of each vertex is $n-1$. A graph is Eulerian if and only if the degree of each vertex is even. Therefore, K_n is Eulerian if n is odd.

(ii) The only semi-Eulerian complete graph is K_2.

14. Find, if possible, an Euler trail or a semi-Euler trail in this graph:

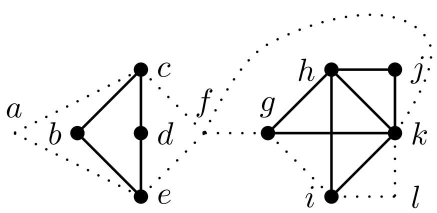


Solution.

The graph is connected, and there are exactly two vertices of odd degree. So there is no Euler trail, but there is a semi-Euler trail beginning at one of the odd vertices, and ending at the other.

Add an edge (fk in the diagram below) between the two vertices of odd degree to make the graph Eulerian, and apply the algorithm using the removal of a cycle to find an Euler trail.

Find a closed trail T, and remove from the graph the edges of T, and any vertices which become isolated as a result. For example, in the following diagram, the trail $T = acfkligfea$ has been chosen. Vertices a, f and l become isolated, and are removed.



Call the subgraph which remains H. Now find an Euler trail in each of the components of H. In this case, it's easy enough to see the following Euler trails:

In the left-hand component: $cdebc$

In the right-hand component: $ghjkihkg$

Now choose any vertex of T, and follow T until we reach a vertex of H. Suppose we start at c. Since c is also a vertex of H, follow the entire Euler trail in the left-hand component of H back to c, then follow T until another vertex of H is reached. In this case, the vertex is k. Follow the Euler trail in the right-hand component of H back to k, and then follow T back to c. We then have the Euler trail

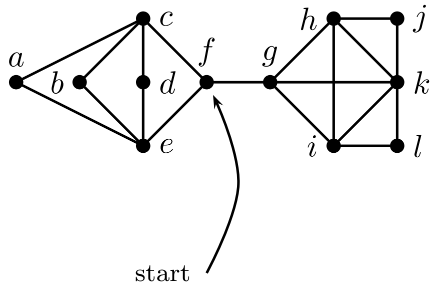
$$cdebcfkihkghjkligfeac.$$

Removing the edge fk we obtain a semi-Euler trail, starting at k and finishing at f:

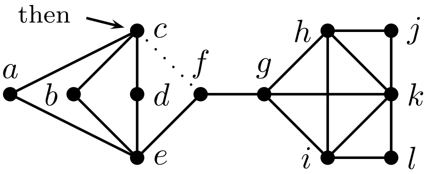
$$kihkghjkligfeacdebcf.$$

Alternate solution

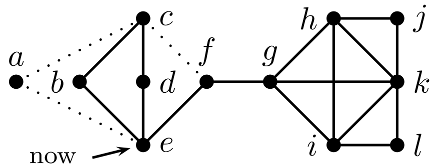
Using Fleury's algorithm, we must start at f or k (f, say), and make a trail, noting at each step which edge is used, and regarding it as being removed from the graph. *However, we must never use an edge, use of which would leave us disconnected from any of the remaining unused edges.* Thus, at step 1, we must not use the edge $[f, g]$. Either $[f, c]$ or $[f, e]$ will do. (This choice means the trail will not be unique.)



Let us choose, say, $[f, c]$. We are then at vertex c, and, having removed edge $[f, c]$, we have (second diagram):



Now we can choose $[c, a]$ or $[c, b]$ or $[c, d]$, none of which disconnect us from any unused (undotted) edges. Choose $[c, a]$, say, in which case we must follow with $[a, e]$. We are now at e (last diagram):



Continuing this way (avoiding such mistakes as now using $[e, f]$, which would leave us disconnected from some unused edges), we will eventually use up all the edges, ending at k. (Note, however, that we will have visited k twice beforehand, and that we will have revisited f once. This is no problem, as a trail being Euler has nothing to do with how many times any *vertices* are used.)

One semi-Euler trail is that which visits the vertices in the following order:

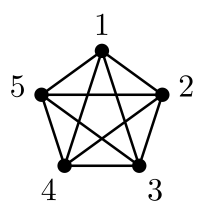
$$fcaebcdefghikjhkgilk.$$

15. (a) A domino is a rectangular tile containing a number of dots (i say) near one end, and a number of dots (j say) near the other end. Call this an $[i, j]$ type domino. Is it possible to arrange 10 dominoes, types $[1, 2]$, $[1, 3]$, $[1, 4]$, $[1, 5]$, $[2, 3]$, $[2, 4]$, $[2, 5]$, $[3, 4]$, $[3, 5]$, $[4, 5]$, end to end so that the numbers of dots on touching ends of adjacent dominoes are always equal?

(b) Repeat the above question in the more general case, when there are $\binom{n}{2}$ dominoes ($n \geq 2$), types all combinations $[i, j]$ with $1 \leq i < j \leq n$.

Solution.

(a) Let G be the graph whose vertices are 1, 2, 3, 4, 5, and whose edges are the domino types (i.e., i is adjacent to j when there is a domino of type $[i, j]$ or $[j, i]$).



Clearly $G = K_5$, the complete graph on 5 vertices (all pairs of vertices adjacent). The question is whether or not an Euler trail (or a semi-Euler trail) exists in G. Since all vertices in G have even degree, there *is* an Euler trail, so the answer is *yes*.

For example, the trail which takes the vertices in the order

$$1, 2, 3, 4, 5, 1, 3, 5, 2, 4, 1$$

is an Euler trail, which uses the edges in the order

$$[1, 2], [2, 3], [3, 4], [4, 5], [5, 1], [1, 3], [3, 5], [5, 2], [2, 4], [4, 1]$$

and which gives an arrangement of the dominoes satisfying the required condition.

(b) Similarly as in part (a), we form K_n, the complete graph on n vertices, which is regular of degree $n-1$ (all vertices having degree $n-1$). Hence a solution certainly exists if $n-1$ is even, i.e., if n is *odd*, since then all vertices have even degree and an Euler trail exists, giving an arrangement of the dominoes in the required way.

If n is 2, there is only one domino, and the result is trivially true — the arrangement is a row containing just that one domino. This corresponds to the fact that K_2 has 2 vertices of odd degree, so a semi-Euler trail exists.

If n is even but greater than 2, K_n has more than 2 vertices of *odd* degree, so no semi-Euler trail exists, and the dominoes cannot be arranged in the required manner.

(Note that this question is equivalent to question **6**.)