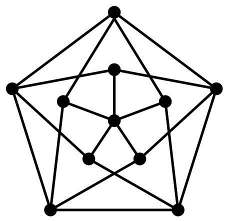
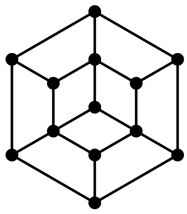
The University of Sydney

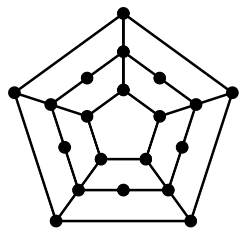
MATH2969/2069

**Graph Theory** **Tutorial 2 (Week 9)** 2008

**1.** Show that the graph on the left is Hamiltonian, but that the other two are not.

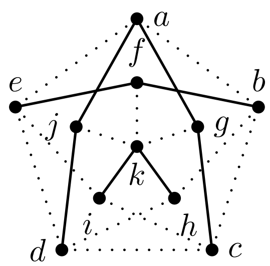




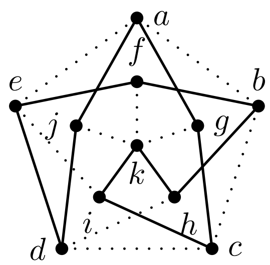


*Solution.*

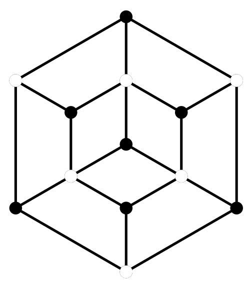
To show that the graph is Hamiltonian, simply find a Hamiltonian cycle. (That is, a cycle which passes through every vertex exactly once.) Unfortunately, finding such a cycle may be tricky, even with so few vertices and edges. It may help to note that exactly two of the edges incident to the central vertex must be used (else this central vertex would not be used exactly once, contrary to requirement). Try assuming that, with the labelling shown, $[k, h]$ and $[k, i]$ are used, and $[k, f]$, $[k, g]$, $[k, j]$ are not. Then, since $f$, $g$, $j$ are to be used, edges $[e, f]$, $[f, b]$, $[a, g]$, $[g, c]$, $[a, j]$, $[j, d]$ must all be used. (See diagram above right.)




With this start, it is not hard to complete a Hamiltonian cycle. An example is shown in the second diagram:




The vertices of the second graph can be coloured, each black or white, in such a way that each edge joins a black vertex to a white vertex (eg., as shown in the diagram). Any path using all vertices would then have to *alternate* between black and white vertices, eg.: $v_1$ = black, $v_2$ = white, $v_3$ = black, $v_4$ = white, ... , $v_{13}$ = black (7 blacks and 6 whites), but this last vertex could not be adjacent with the first to complete the cycle, both having the same colour. Hence no Hamiltonian cycle exists.



In the third graph, there is an outer pentagon (5 sided figure) of 5 vertices, an innermost pentagon of 5 vertices, and an in-between decagon (10 sided figure) of 10 vertices.

On the decagon, 5 of the vertices are of degree 2, which means that in any Hamiltonian cycle all 10 edges of this decagon must be used. (For a path to go through a vertex of degree 2, both incident edges must be used.)

Since each vertex must be used only once, this means that none of the edges directed radially outwards from the centre of the graph as drawn can be used. That is, none of the edges connecting the outer pentagon, the decagon, and the inner pentagon can be used. So we cannot get from either pentagon to the decagon.

Hence no Hamiltonian cycle can exist.

**2.** Is it possible for a simple connected graph containing a bridge to be Hamiltonian?

*Solution.*

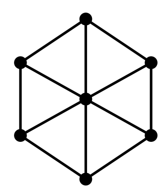
A bridge is an edge whose removal would disconnect the graph. Clearly, any Hamiltonian cycle in a graph with a bridge would have to include the bridge. But a bridge is not part of any cycle (since if it were, its removal would not disconnect the graph). So any graph with a bridge is not Hamiltonian.

**3.** What is wrong with the following argument?

Suppose $G$ is a simple graph with degree sequence $(3, 3, 3, 3, 3, 3, 6)$. Let $u$ and $v$ be two non-adjacent vertices with degree 3, so that $\deg(u)+\deg(v) = 6 \leq 7$. Since the number of vertices is 7, it follows by Ore's Theorem that $G$ is not Hamiltonian.

*Solution.*

This is an incorrect use of Ore's Theorem, which states that if $G$ is a simple graph with $n \geq 3$ vertices and $\deg(u)+\deg(v) \geq n$ for each pair of non-adjacent vertices $u$ and $v$, then $G$ is Hamiltonian. The theorem cannot be used to prove that a graph is not Hamiltonian. Indeed, a graph with degree sequence $(3, 3, 3, 3, 3, 3, 6)$ is as shown, and it is easy to see that it is Hamiltonian.

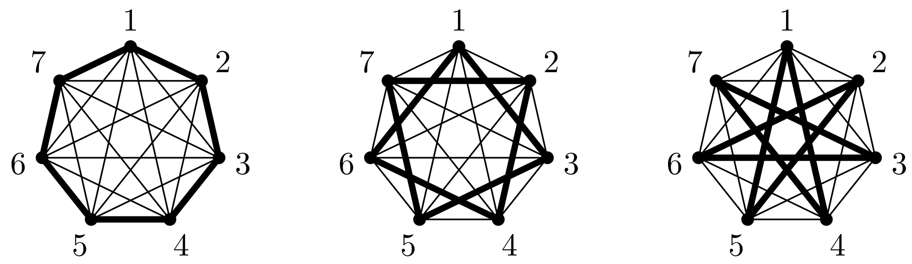


**4.** A club has seven members and meets for lunch each month. If the members sit at a round table and decide to sit so that each member has different neighbours at each lunch, determine how many months this arrangement could last, and give possible seating arrangements for these months.

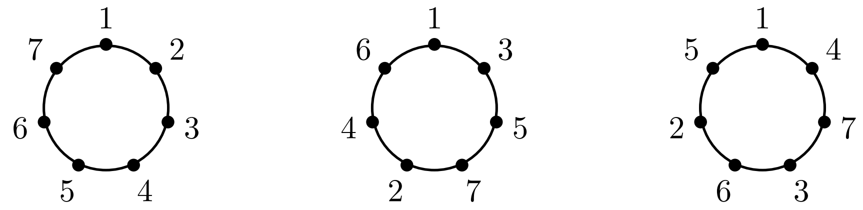
*Solution.*

A seating arrangement for one month is given by a Hamiltonian cycle in the complete graph whose vertices are the seven members ($K_7$). To keep the desired arrangement going as long as possible, we need as many *disjoint* (i.e., edge-disjoint) Hamiltonian cycles as possible.

Each Hamiltonian cycle uses 7 edges, out of all $\binom{7}{2} = \frac{7\times 6}{2} = 21$ edges in $K_7$. Hence there are at most $21/7 = 3$ disjoint Hamiltonian cycles. To show that there are actually 3, we need only find them:




Corresponding seatings round the table:




**5.** (*i*) How many edges does the complete bipartite graph, $K_{m,n}$, have?

(*ii*) How many complete bipartite graphs have $k$ vertices?

(*iii*) What is the maximum number of edges in a simple bipartite graph with $k$ vertices?

*Solution.*

(*i*) The vertex set of $K_{m,n}$ consists of two disjoint sets, $A$ and $B$, say, such that: $A$ contains $m$ vertices and $B$ contains $n$ vertices; each vertex in $A$ is adjacent to each vertex in $B$; no two vertices in $A$, or in $B$, are adjacent. Hence, the degree of each vertex in $A$ is $n$, and the degree of each vertex in $B$ is $m$. Therefore, the sum of the degrees is $2 \times m \times n$, and there are $m \times n$ edges.

(*ii*) If $K_{m,n}$ has $k$ vertices, then $m + n = k$, and $n = k - m$. So the question is, how many complete bipartite graphs are there of the type $K_{m,k-m}$? The possible values for $m$ are $1, 2, \ldots, k-1$. But $K_{m,k-m}$ is, of course, isomorphic to (that is, is the same as) $K_{k-m,m}$.

If $k$ is an even number, the values of $m$ which will produce non-isomorphic (that is, different) graphs are $1, 2, \ldots, k/2$. So there are $k/2$ complete bipartite graphs with $k$ vertices – namely, $K_{1,k-1}$, $K_{2,k-2}, \ldots, K_{k/2,k/2}$.

If $k$ is odd, the values of $m$ which produce non-isomorphic graphs are $1, 2, \ldots, (k-1)/2$, and so there are $(k-1)/2$ graphs – namely,
$K_{1,k-1}$, $K_{2,k-2}, \ldots, K_{(k-1)/2,(k+1)/2}$.

(*iii*) Clearly, the maximum number of edges occurs when the graph is complete. So we are looking for the value of $m$ for which $K_{m,k-m}$ has the most edges. By part (*i*) the number of edges in $K_{m,k-m}$ is $m(k-m)$. This function of $m$ (whose graph is an upside down parabola), has a maximum value at $m = k/2$, at which $m(k-m) = k^2/4$. That is, the maximum number of edges in a bipartite graph with $k$ vertices is $k^2/4$.

(This maximum is only achieved if $k$ is even. If $k$ is odd, the maximum number of edges is $\frac{k-1}{2} \times \frac{k+1}{2} = \frac{k^2-1}{4}$.)

**6.** (*i*) How many vertices does the $k$-cube graph, $Q_k$, have?

(*ii*) What is the degree of each vertex in $Q_k$?

(*iii*) How many edges does $Q_k$ have?

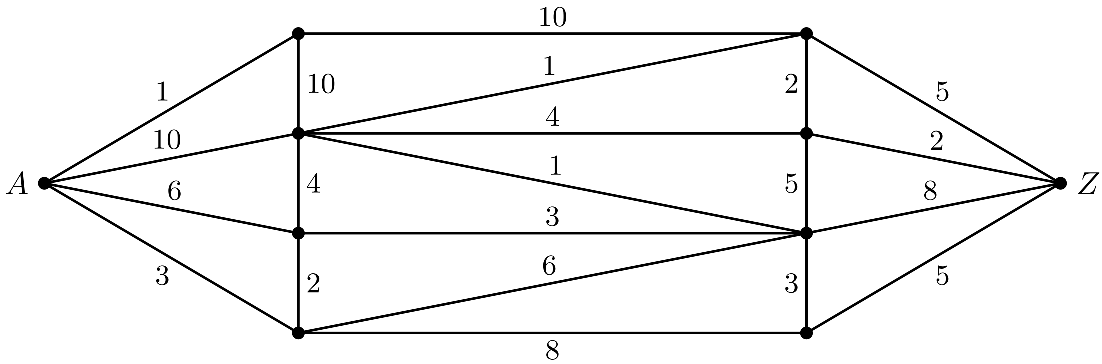
*Solution.*

(*i*) In $Q_k$ vertices correspond to the sequences $(a_1, a_2, \ldots, a_k)$, where each $a_i = 0$ or 1. So the number of vertices is equal to the number of such sequences. Since the sequence is of length $k$, and there are two choices for each place in the sequence, there are $2^k$ sequences, and hence $2^k$ vertices.

(*ii*) An edge joins vertices if the sequences corresponding to the vertices differ in exactly one place. Consider the sequence corresponding to any particular vertex. Since it is of length $k$, there are $k$ sequences which differ from it in exactly one place. So the degree of any vertex is $k$.

(*iii*) From parts (*i*) and (*ii*) the sum of the degrees is $k \times 2^k$. Therefore the number of edges is $\frac{k \times 2^k}{2} = k \times 2^{k-1}$.

**7.** In the following graph, find all shortest paths from $A$ to $Z$.




*Solution.*

While a short path, from $A$ to $Z$, maybe even the shortest, may be found by inspection, the following algorithm guarantees that we really do find the shortest.

Start by giving $A$ the permanent label [0]. Label each vertex adjacent to $A$ with a temporary label equal to the distance from $A$ to the vertex.

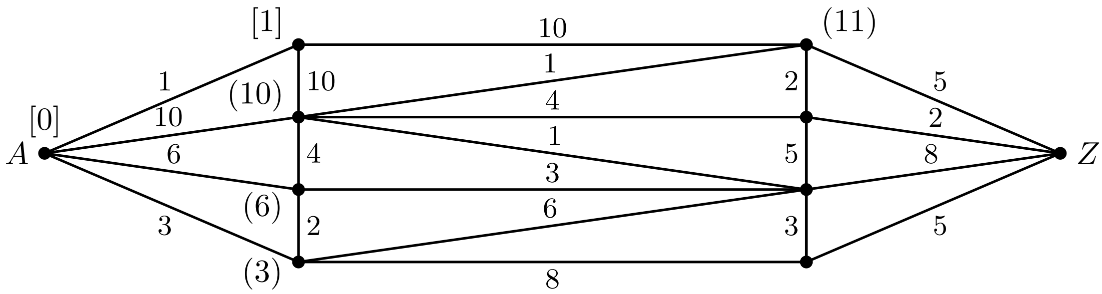
Find the smallest temporary label (in this case (1)) and make it permanent. If the vertex which has just been made permanent is $V$, look at all vertices adjacent to $V$.

If an adjacent vertex has no temporary label, give it one equal to the permanent label on $V$ plus the distance from $V$.

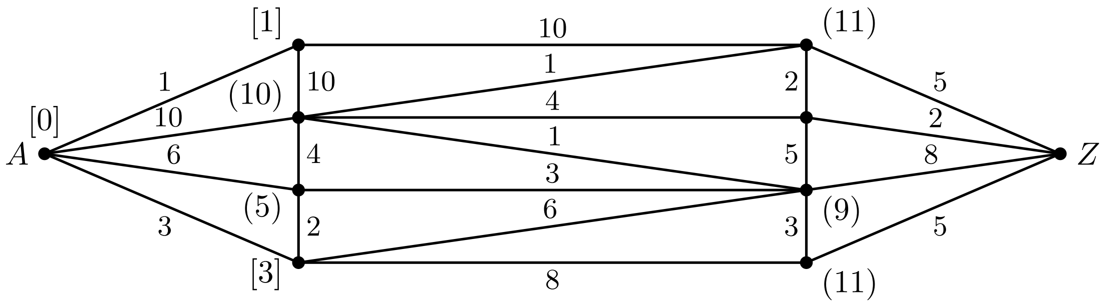
If an adjacent vertex is already labelled permanently, leave it alone.

If an adjacent vertex is already labelled temporarily, and the label is bigger than the permanent label on $V$ plus the distance from $V$, reduce the temporary label so that it equals the permanent label on $V$ plus the distance from $V$. Otherwise leave it alone.

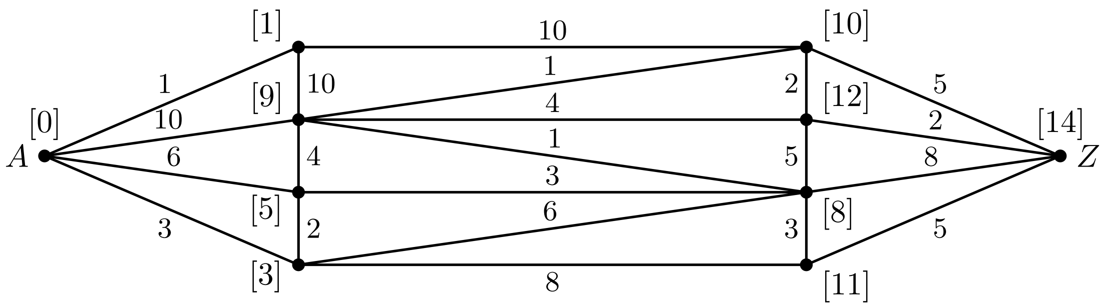
Having labelled all vertices adjacent to the vertex labelled [1] we have the following diagram (where round brackets have been used for temporary labels, and square brackets for permanent labels) :




Now find the next smallest temporary label (in this case (3)), and repeat the process described above. The following diagram results after making the label (3) permanent, and labelling all adjacent vertices. (Note that the temporary label (6) has been reduced to (5).)

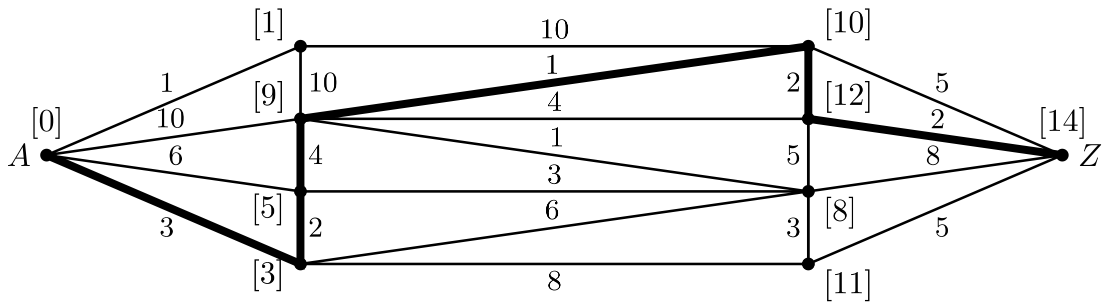


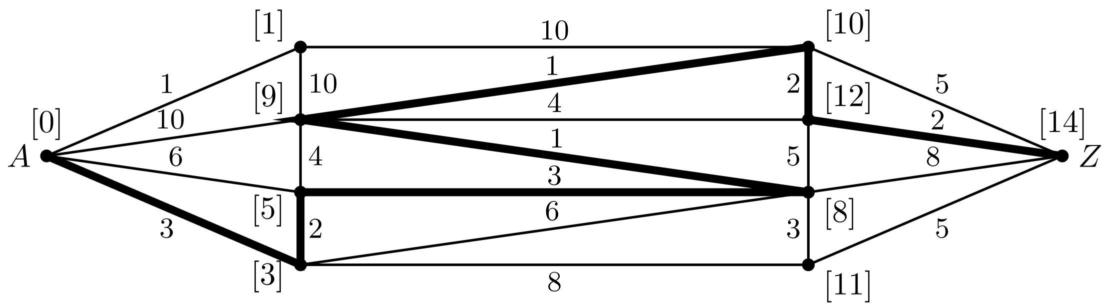

Repeat the process until all vertices have permanent labels, resulting in the following diagram:



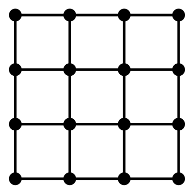

The label on each vertex is now the shortest distance from $A$ to that vertex, and, tracing back from $Z$, we can find all shortest paths from $A$ to $Z$.

Since $14-10<5$, $\quad 14-12=2$, $\quad 14-8<8$, $\quad 14-11<5$, backtrack from $Z$ to the vertex labelled [12].
Now $12-10=2$, $\quad 12-10<4$, $\quad 12-8<5$, and so backtrack from [12] to [10].
Continuing in this way, we find *two* shortest paths, of length 14, from $A$ to $Z$ (because from [9] we can equally well backtrack to [5] or to [8]):



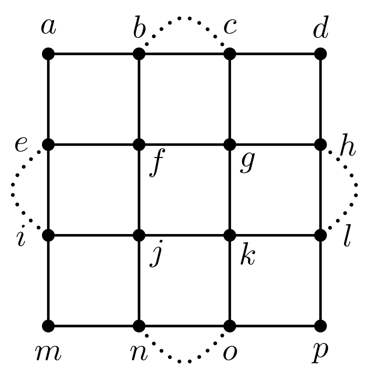




**8.** Find a solution to the Chinese Postman Problem in this graph, given that every edge has equal weight.



*Solution.*

The problem is to find the shortest closed walk which uses every edge at least once. The graph is not Eulerian and so some edges will have to be duplicated in order to find a closed walk using each edge at least once. Since there are 8 vertices of odd degree in the graph at least 4 edges will have to be duplicated, and since the 8 vertices are adjacent to one another in pairs, it is easy to see that a solution results if we add the 4 dotted edges as shown below.



Any Euler trail in this graph is a solution.
One such trail is $abcghlkjiefgkonjfbcdhlponmiea$.

**9.** (*i*) Prove that a graph in which the degree of each vertex is at least two contains a cycle.

(*ii*) Prove that a tree with at least 2 vertices has at least 2 vertices of degree 1.

*Solution.*

(*i*) The result is trivial if the graph has loops or multiple edges, so suppose the graph is simple. Construct a walk $v_1 \to v_2 \to v_3 \dots$ in the graph by choosing any vertex as $v_1$, any vertex adjacent to $v_1$ as $v_2$, any vertex adjacent to $v_2$ other than $v_1$ as $v_3$, and so on. That is, at each vertex $v_i$, choose $v_{i+1} \neq v_{i-1}$. The existence of such a vertex is guaranteed by the fact that the degree of each vertex is at least 2. Since the graph has only finitely many vertices, at some stage a vertex will be repeated. Then there is a cycle between the two occurrences of the repeated vertex.

(*ii*) Note that part (*i*) is equivalent to:
A graph with no cycles (and at least one edge) has at least one vertex with degree 1.

Since a tree is defined as a connected graph without cycles, it follows that a tree has at least one vertex of degree 1. Now construct a path as follows:

Find a vertex of degree 1, and move to its adjacent vertex. If that vertex has degree 1, then we have found 2 vertices of degree 1. Otherwise, its degree is at least 2, and we can choose another edge along which to move to a third vertex. Keep doing this, going along previously unused edges as long as possible.

Since there are no cycles, we will never get back to an already visited vertex, and since there are only finitely many vertices, we cannot keep getting new vertices forever. Therefore, we must eventually reach a new vertex from which we are unable to continue, i.e., a vertex not previously visited , and with no unused edge incident to it. That is, we have found a second vertex of degree 1.

Here is an alternative proof:

Suppose there are $n > 1$ vertices, and $p$ of them have degree 1. Hence the other $n-p$ vertices have degree $> 1$, i.e., $\geq 2$ (since the tree is connected, with at least two vertices, so no vertex has degree 0). Now recall that a tree with $n$ vertices has $n-1$ edges.

By the Handshaking lemma,

$$\begin{aligned} 2(\text{number of edges}) = 2(n-1) &= (\text{total degree}) \\ &\geq 1 \times p + 2(n-p) = 2n - p, \\ \text{which gives} \quad & p \geq 2. \end{aligned}$$

**10** Prove, directly from the definition of a tree as a connected graph without cycles, that the additon of one edge to a tree creates exactly one cycle.

*Solution.*

Firstly, since a tree has no cycles it cannot have multiple edges or loops, and so there is at most one edge between any pair of vertices.

If an edge is added to a pair of adjacent vertices, then exactly one cycle is formed, consisting of the the one existing edge and the added edge.

Now suppose that an edge is added to a pair of non-adjacent vertices, $v_i$ and $v_j$. Since a tree is connected there is a path from $v_i$ to $v_j$, and so adding the edge $v_i v_j$ forms the cycle $v_i \dots v_j v_i$. More than one cycle could be formed only if there were at least two paths from $v_i$ to $v_j$. But if there were more than one path between $v_i$ and $v_j$ in the tree, then there would be a cycle. Hence, there is exactly one path between any pair of vertices, and exactly one cycle created by adding an edge.

**11.** How many edges are there in a forest with $v$ vertices and $k$ components?

*Solution.*

Suppose the number of vertices in the $i$th component of the forest is $v_i$. Each component of a forest is a tree, and so the number of edges in the $i$th component is $v_i - 1$. The total number of edges is therefore

$$\sum_{i=1}^{k}(v_i - 1) = \sum_{i=1}^{k} v_i - \sum_{i=1}^{k} 1 = v - k.$$

**12.** Let $T$ be a tree with $p$ vertices of degree 1 and $q$ other vertices. Show that the sum of the degrees of the vertices of degree greater than 1 is $p + 2(q-1)$.

*Solution.*

The number of vertices is $p + q$, and so the number of edges is $e = p + q - 1$. Hence, by the Handshaking Lemma,

$$\begin{aligned} \text{Total degree} &= \sum_{v_i} \deg(v_i) \\ &= p + \sum_{\deg(v_i) > 1} \deg(v_i) \\ &= 2e \\ &= 2(p + q - 1), \end{aligned}$$

and so $$\sum_{\deg(v_i) > 1} \deg(v_i) = 2(p + q - 1) - p = p + 2(q-1).$$

**13.** Show that if a tree has two vertices of degree 3, then it must have at least 4 vertices of degree 1.

*Solution.*

Let $v$ be the number of vertices, and $p$ the number of vertices of degree 1. If two vertices have degree 3, there are $(v - p - 2)$ vertices with degree other than

1 or 3. In particular, each of these $(v - p - 2)$ vertices has degree at least 2.

$$
\begin{aligned}
\sum \text{degrees} &\geq p + 2 \times 3 + 2(v - p - 2) \\
&= 2v - p + 2. \\
\text{But} \quad \sum \text{degrees} &= 2 \times \text{number of edges} \\
&= 2(v - 1). \\
\text{Therefore} \quad 2v - 2 &\geq 2v - p + 2, \\
\text{or} \quad p &\geq 4.
\end{aligned}
$$

**14.** Show that, for each value of $n \geq 1$, the graph associated with the alcohol molecule $C_nH_{2n+1}OH$ is a tree. (Carbon, Hydrogen, Oxygen have valencies 4, 1, 2 respectively.)

*Solution.*

Since a molecule is a collection of atoms connected by chemical bonds, the associated graph is connected. It consists of $n$ vertices of degree 4 each labelled $C$, $2n+2$ vertices of degree 1 each labelled $H$, and 1 vertex of degree 2 labelled $O$.

Hence the number of vertices is $v = n + (2n + 2) + 1 = 3n + 3$, and, using the Handshaking Lemma, the number of edges is $e = \frac{1}{2}\big(4n+1(2n+2)+2\big) = 3n+2$. Since $e = v - 1$, and the graph is connected, it must be a tree.

**15.** $(i)$ Find the number of molecules with formula $C_5H_{12}$, and draw them.

$(ii)$ How many non-isomorphic trees are there with 5 vertices?

$(iii)$ Comment on the relationship between your answers to parts $(i)$ and $(ii)$.

*Solution.*

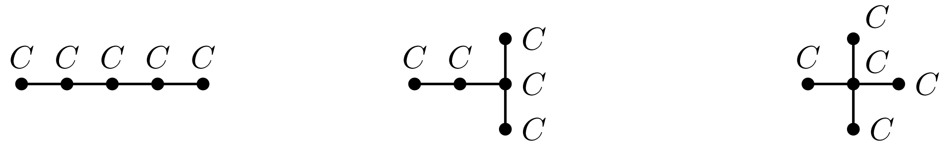
$(i)$ As in the previous question, the graph associated with any such molecule is connected, the number of vertices is 17, and the number of edges is $\frac{4\times5+12}{2} = 16$. So the graph is a tree, and therefore must be simple. (So, for example, there can be no double bonds like $\cdots C = C \cdots$.) Moreover, each $C$ must be adjacent to at least one other $C$, else we would have a component of the graph of the form

$$
\begin{array}{ccccc}
 & & H & & \\
 & & | & & \\
H & — & C & — & H \\
 & & | & & \\
 & & H & &
\end{array}
$$

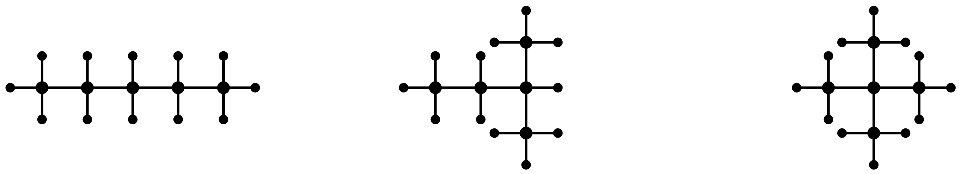
disconnected from the rest of the graph (since each $H$ has valency only 1).

Hence the $C$s and the edges between them must form a connected subgraph — a subtree in fact.

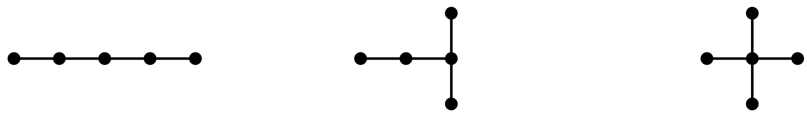
The 5 carbon atoms could be connected up with single bonds in three essentially different (non-isomorphic) ways:




In each case there is an essentially unique way to add the $H$s, giving the following 3 isomers of $C_5H_{12}$ (where $C = \bullet$, $H = \cdot$ ):



($ii$) There are 3 non-isomorphic trees on 5 vertices:

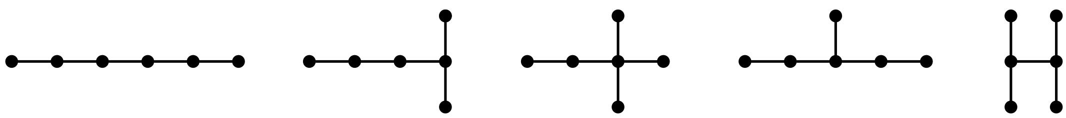


($iii$) The 3 trees in part ($ii$) are precisely those formed by the carbon atoms in part ($i$).

**16.** Repeat question **7** for $C_6H_{14}$, and trees with 6 vertices.

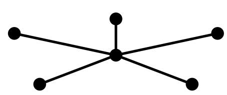
*Solution.*

($i$) There are 5 isomers of $C_6H_{14}$, given by the following 5 arrangements of the carbon atoms:



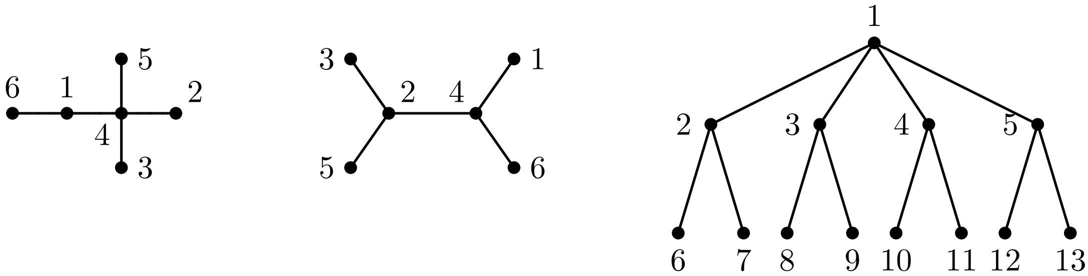
Adding the $H$s in the obvious way gives the 5 molecules.

($ii$) There are 6 non-isomorphic trees on 6 vertices – the 5 shown in part ($i$), and the following:



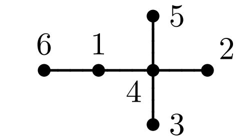
($iii$) In this case, the 6 carbon atoms can be arranged in a way which corresponds to only 5 of the 6 non-isomorphic trees on 6 vertices. (In the sixth tree, one vertex has degree 5, and carbon has valency 4.)

**17.** Find the Prüfer sequence corresponding to each of the following labelled trees:

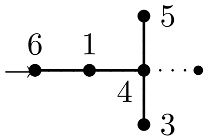


*Solution.*

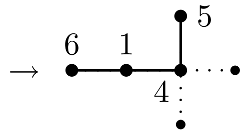
We find the lowest labelled leaf (vertex of degree 1), delete that leaf and its incident edge, and write down the label of the vertex to which the just deleted leaf was adjacent. Continue until just two vertices are left.



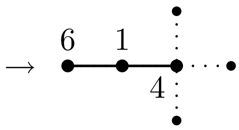
Lowest leaf: 2
Remove 2, write 4



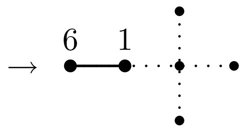
Lowest leaf: 3
Remove 3, write 4



Lowest leaf: 5
Remove 5, write 4



Lowest leaf: 4
Remove 4, write 1



Two left: Stop.
Answer: (4,4,4,1)

Similarly, with the second graph: remove 1 and write 4; remove 3 and write 2; remove 5 and write 2; remove 2 and write 4; stop. The sequence is (4,2,2,4).

For the third graph the sequence is $(2, 2, 1, 3, 3, 1, 4, 4, 1, 5, 5)$.

**18.** Find the labelled trees corresponding to these Prüfer sequences:

$(i)$ $(1, 2, 3, 4, 5)$ $(ii)$ $(3, 3, 3, 3, 3)$ $(iii)$ $(2, 8, 6, 3, 1, 2)$

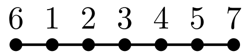
*Solution.*

$(i)$ In order to find the tree corresponding to a particular Prüfer sequence $(a_1, a_2, \ldots, a_{n-2})$ (where each $a_i$ is one of 1,2, … ,$n$), we augment the sequence to $(a_1, a_2, \ldots, a_{n-2}, n)$ , and form a new sequence $(b_1, b_2, \ldots, b_{n-2}, b_{n-1})$, term by term, such that each $b_i$ is the smallest positive integer not equal to any of $b_1,\ b_2,\ \ldots,\ b_{i-1},\ a_i,\ a_{i+1},\ \ldots,\ a_{n-2}, n$. The labelled graph whose edges are $[a_1, b_1]$, $[a_2, b_2]$, … , $[a_{n-2}, b_{n-2}]$, $[b_{n-1}, n]$ is then the labelled tree with Prüfer sequence $(a_1, a_2, \ldots, a_{n-2})$.

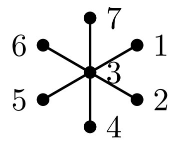
From (1,2,3,4,5), (5 terms, so $n = 7$) we augment to (1,2,3,4,5,7) then form (6,1,2,3,4,5), as follows:

the least positive integer not equal to any of 1,2,3,4,5,7 is 6;
the least positive integer not equal to any of 2,3,4,5,7 or 6 is 1;
the least positive integer not equal to any of 3,4,5,7 or 6,1 is 2;
the least positive integer not equal to any of 4,5,7 or 6,1,2 is 3;
the least positive integer not equal to any of 5,7 or 6,1,2,3 is 4;
the least positive integer not equal to any of 7 or 6,1,2,3,4 is 5.

Hence the edges are [1,6], [2,1], [3,2], [4,3], [5,4], [7,5]:




(*ii*) Similarly, from (3,3,3,3,3), again $n = 7$, so we augment to (3,3,3,3,3,7) then form (1,2,4,5,6,3): edges [3,1], [3,2], [3,4], [3,5], [3,6], [7,3].




(*iii*) With (2,8,6,3,1,2) (6 terms), $n = 8$, so we augmentto (2,8,6,3,1,2,8) then form (4,5,7,6,3,1,2):edges [2,4], [8,5], [6,7], [3,6], [1,3], [2,1], [8,2].