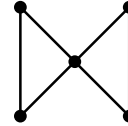
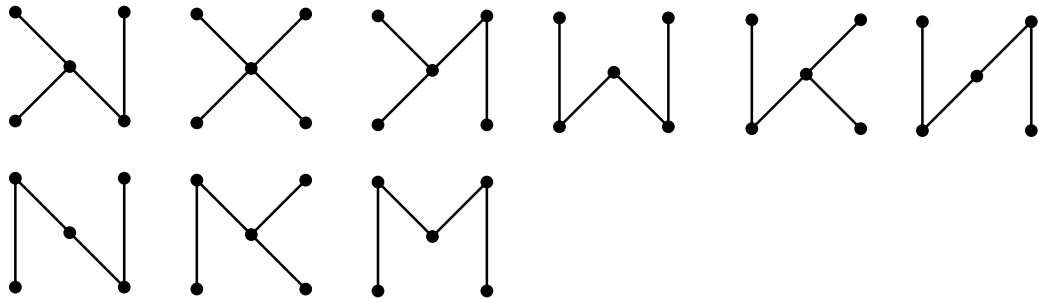


1. Draw all the spanning trees of this graph:



Solution.

Since the graph has 5 vertices, a spanning tree will have 4 edges. So two edges have to be removed, and it is clear that exactly one edge from each of the cycles must be removed. There are 9 spanning trees, as shown:



2. If A is the adjacency matrix of a simple graph G with vertex set $\{v_1, v_2, \dots, v_n\}$, and $(A^k)_{ij}$ is the (i, j) term of A^k , show that $(A^2)_{ii} = \deg(v_i)$.

Solution.

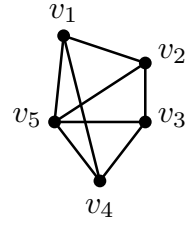
Recall that if $A = (a_{ij})$ is the adjacency matrix of a graph G with n vertices v_1, v_2, \dots, v_n , then A is an $n \times n$ symmetric matrix, and a_{ij} is the number of edges joining v_i, v_j (0 or 1 for a simple graph), i.e., the number of walks of length 1 (i.e., using 1 edge) from v_i to v_j . More generally, $(A^k)_{ij}$ is the number of walks of length k (k steps along edges) from v_i to v_j . (Walks allow the use of the same edge more than once.)

$(A^2)_{ii}$ is the number of walks of length 2 starting at v_i and finishing at v_i . In a simple graph, you can only get back to the starting vertex in two steps by going to an adjacent vertex and back, i.e., out along an edge incident to v_i and back again along the same edge. Hence the number of ways is the degree of v_i .

Or, from first principles,

$$\begin{aligned} (A^2)_{ii} &= a_{i1}a_{1i} + a_{i2}a_{2i} + \dots + a_{in}a_{ni} \\ &= a_{i1}^2 + a_{i2}^2 + \dots + a_{in}^2 \\ &= (0 \text{ or } 1) + (0 \text{ or } 1) + \dots + (0 \text{ or } 1) \\ &= \text{the number of vertices } v_j \text{ adjacent to } v_i \\ &= \text{the degree of } v_i. \end{aligned}$$

3. (i) Write down the adjacency matrix, A , for this graph.
- (ii) Calculate A^2 , and verify that the result proved in Question 2 holds.
- (iii) Find the number of different walks of length 4 from v_5 to v_5 .
- (iv) Verify that the trace of A^3 is 6 times the number of triangles in the graph.



Solution.

$$(i) \quad A = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \quad (ii) \quad A^2 = \begin{pmatrix} 3 & 1 & 3 & 1 & 2 \\ 1 & 3 & 1 & 3 & 2 \\ 3 & 1 & 3 & 1 & 2 \\ 1 & 3 & 1 & 3 & 2 \\ 2 & 2 & 2 & 2 & 4 \end{pmatrix}$$

In order to verify that the result in Question 2 holds here, check that the entries on the main diagonal of A^2 – that is, 3, 3, 3, 3, 4 – are the degrees of v_1, v_2, v_3, v_4 and v_5 respectively.

- (iii) The number of different walks of length 4 from v_5 to v_5 is the $(5, 5)$ entry in A^4 . The $(5, 5)$ entry in A^4 is equal to the dot product of row 5 of A^2 and column 5 of A^2 . That is, $(2 \ 2 \ 2 \ 2 \ 4) \cdot (2 \ 2 \ 2 \ 2 \ 4) = 32$.
- (iv) There are 4 triangles in the graph. The trace of A^3 is the sum of its diagonal entries. Multiplying A^2 by A we find that the entries on the main diagonal of A^3 are 4, 4, 4, 4, 8, so that the trace of $A^3 = 4 + 4 + 4 + 4 + 8 = 24 = 6 \times 4$. (Note that a triangle in a graph is a walk of length 3 from a vertex to itself. The element $(A^3)_{ii}$ on the main diagonal of A^3 is the number of walks of length 3 from vertex i to itself. Adding all the diagonal elements counts each triangle 6 times, and so the number of triangles in a graph (without loops) is $\frac{1}{6} \times$ the trace of A^3 .)

4. A graph G has adjacency matrix $A = \begin{pmatrix} 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$.

- (i) Is G a simple graph? (ii) What is the degree sequence of G ?
- (iii) How many edges does G have?

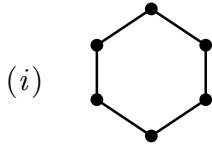
Solution.

- (i) Since the matrix contains entries other than 1s and 0s, G is not simple. For example, there are 2 edges from vertex 1 to vertex 4.
- (ii) The sum of the entries in any row is the degree of the vertex corresponding to that row. The degree sequence is therefore $(2, 2, 3, 3, 4)$.
- (iii) The sum of the degrees is 14, and so G has 7 edges.
5. Let A be the adjacency matrix of a bipartite graph. Prove that the diagonal entries of A^{2n+1} are all equal to 0, for any natural number n .

Solution.

If a graph is bipartite, then its vertices can be partitioned into two sets, V_1 and V_2 say, such that each edge in the graph joins a vertex in V_1 to one in V_2 . Any walk alternates between vertices in V_1 and V_2 , and so any walk which starts and ends at the same vertex must use an even number of edges. That is, there are no walks of odd length from a vertex to itself in a bipartite graph. A diagonal entry of A^{2n+1} is equal to the number of walks of length $2n+1$ from some vertex to itself, and since $2n+1$ is odd, all such entries are 0.

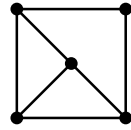
6. Find the number of spanning trees in each of the following graphs:



(ii) K_7

(iii) $K_{3,3}$

(iv)



Solution.

- (i) There are 6 vertices, and so a spanning tree has 5 edges. Deleting any one of the edges gives a spanning tree, and since there are 6 edges to choose from there are 6 spanning trees.
- (ii) The number of spanning trees of the complete graph K_7 is equal to the number of labelled trees on 7 vertices. By Cayley's Theorem, there are 7^5 labelled trees on 7 vertices, and so there are $7^5 = 16807$ spanning trees of K_7 .
- (iii) In this part we use the *Matrix Tree Theorem*: Let M be the matrix obtained from the adjacency matrix A of a simple graph G by changing all the 1s in A to -1 s, and by replacing each 0 on the main diagonal with the degree of the corresponding vertex. Then the number of spanning trees of G is equal to the value of any cofactor of M .

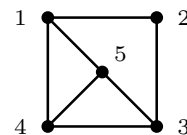
For $K_{3,3}$, the adjacency matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and so} \quad M = \begin{pmatrix} 3 & 0 & 0 & -1 & -1 & -1 \\ 0 & 3 & 0 & -1 & -1 & -1 \\ 0 & 0 & 3 & -1 & -1 & -1 \\ -1 & -1 & -1 & 3 & 0 & 0 \\ -1 & -1 & -1 & 0 & 3 & 0 \\ -1 & -1 & -1 & 0 & 0 & 3 \end{pmatrix}.$$

The (i, j) -cofactor of M is $(-1)^{i+j}$ times the determinant of the submatrix of M obtained by deleting row i and column j . We may use any cofactor to calculate the number of spanning trees. The $(1, 1)$ cofactor of M is

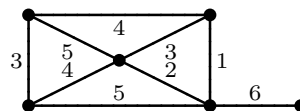
$$\begin{aligned}
& (-1)^{1+1} \begin{vmatrix} 3 & 0 & -1 & -1 & -1 \\ 0 & 3 & -1 & -1 & -1 \\ -1 & -1 & 3 & 0 & 0 \\ -1 & -1 & 0 & 3 & 0 \\ -1 & -1 & 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 & -1 & -1 \\ 0 & 3 & -1 & -1 & -1 \\ 1 & -1 & 3 & 0 & 0 \\ 1 & -1 & 0 & 3 & 0 \\ 1 & -1 & 0 & 0 & 3 \end{vmatrix} \quad \begin{array}{l} \text{(replacing column 1 by} \\ \text{column 1 + column 2 +} \\ \dots + \text{column 5)} \end{array} \\
& = \begin{matrix} (R3=R3-R5) \\ (R4=R4-R5) \end{matrix} \begin{vmatrix} 0 & 0 & -1 & -1 & -1 \\ 0 & 3 & -1 & -1 & -1 \\ 0 & 0 & 3 & 0 & -3 \\ 0 & 0 & 0 & 3 & -3 \\ 1 & -1 & 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 0 & -1 & -1 & -1 \\ 3 & -1 & -1 & -1 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 3 & -3 \end{vmatrix} = -3 \begin{vmatrix} -1 & -1 & -1 \\ 3 & 0 & -3 \\ 0 & 3 & -3 \end{vmatrix} \\
& = \text{(eventually) } 81.
\end{aligned}$$

- (iv) Using the Matrix Theorem again,
with the vertices labelled as shown,



$$M = \begin{pmatrix} 3 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{pmatrix}, \text{ and the } (1,1)\text{-cofactor is } \begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 3 & -1 \\ 0 & -1 & -1 & 3 \end{vmatrix} = 24.$$

7. Use Kruskal's algorithm to find all least weight spanning trees for this weighted graph.

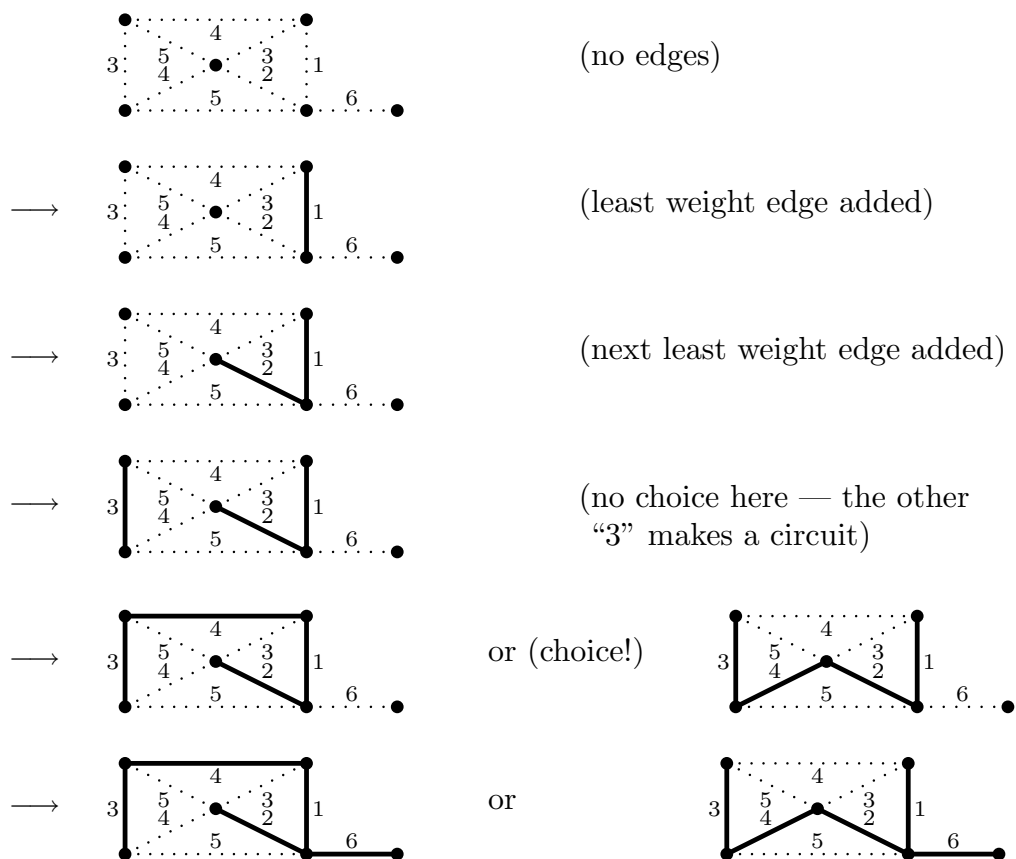


Solution.

A spanning tree is a subgraph which is a tree (i.e., is connected, but contains no circuits) and which involves all the vertices. We need the spanning trees which minimise the total weight of the edges involved (the weight of each edge being given in the diagram).

Kruskal's algorithm is as follows: Starting with the subgraph consisting of all the vertices but no edges, we keep adding a least weight edge from those edges which are as yet unused and do not make a circuit with those already selected. We terminate when a spanning tree is formed.

There may be a choice of equally low weight edges at some stages, which may (or may not) result in more than one minimal weight spanning tree being constructable. However, whichever choice is made, a minimal weight spanning tree will result. Moreover all minimal weight spanning tree can be obtained by making suitable choices.



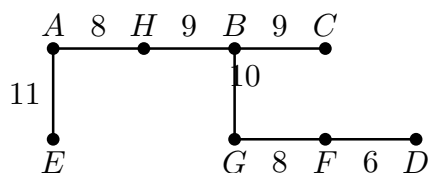
We now have all the least weight spanning trees — only two in this case.

8. Find a minimum cost spanning tree for the graph with this cost matrix. How many such trees are there?

	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	<i>G</i>	<i>H</i>
<i>A</i>	0	12	0	14	11	0	17	8
<i>B</i>	12	0	9	0	12	15	10	9
<i>C</i>	0	9	0	18	14	31	0	9
<i>D</i>	14	0	18	0	0	6	23	14
<i>E</i>	11	12	14	0	0	15	16	0
<i>F</i>	0	15	31	6	15	0	8	16
<i>G</i>	17	10	0	23	16	8	0	22
<i>H</i>	8	9	9	14	0	16	22	0

Solution.

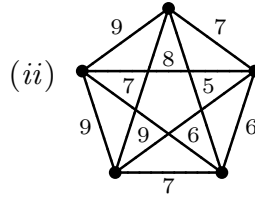
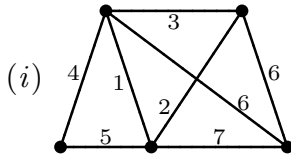
Kruskal's algorithm may be used directly from the cost matrix, without necessarily drawing a diagram of the graph. Simply choose the least cost edges, making sure that no cycle is formed, until seven edges have been chosen. One minimum cost spanning tree is:



Note that after choosing the edge with cost 6, and the two edges with costs 8, the next smallest cost edges are BH , CH and BC , all with cost 9. Since these edges form a triangle, only two of them may be used. Once two have been chosen, there are no further choices to be made, since there is just one edge with cost 10, and one with cost 11. There are three possible choices of two edges from three, and so there are 3 minimum cost spanning trees.

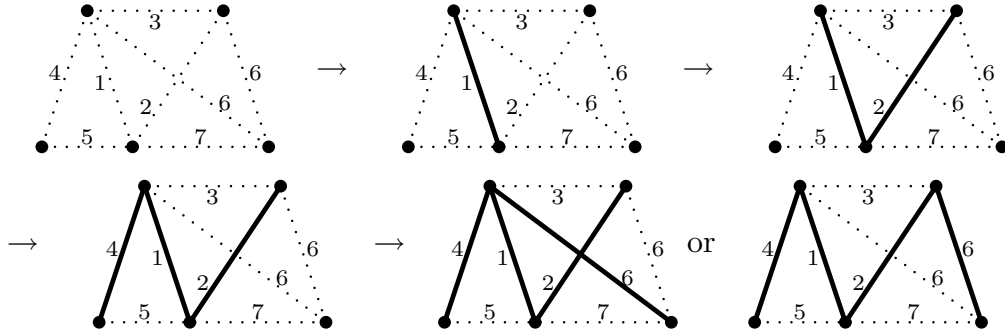
Extra questions

9. Find a minimum weight spanning tree in each of the following weighted graphs:



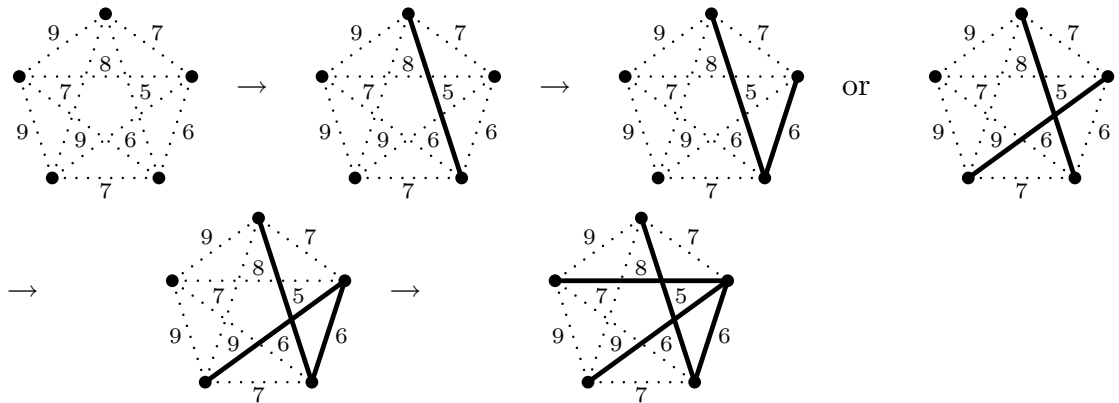
Solution.

(i)



There are two solutions, because of the choice of “6”s at the last step. Note that “3” and “5”, when they were the least weight unused edge, were not allowed, as each would make a circuit with already used edges.

(ii)



Although there was a choice of “6”s at the second step, both were eventually used anyway, so there was only one solution. Note that use of any of the “7”s would have made a circuit.

10. Use the Matrix Tree Theorem to verify the fact that the number of spanning trees of the complete graph K_n is equal to the number of labelled trees on n vertices.

Solution.

The adjacency matrix of K_n is an $n \times n$ matrix with 0s on the diagonal, and a 1 in every other position. The degree of every vertex is $n - 1$, and so the matrix M we use to find the number of spanning trees is the $n \times n$ matrix

$$M = \begin{pmatrix} n-1 & -1 & -1 & \dots & -1 \\ -1 & n-1 & -1 & \dots & -1 \\ -1 & -1 & n-1 & \dots & -1 \\ -1 & -1 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & -1 \\ -1 & -1 & -1 & \dots & n-1 \end{pmatrix}.$$

Taking the (1,1) cofactor, the number of spanning trees is:

$$\begin{vmatrix} n-1 & -1 & -1 & \dots & -1 \\ -1 & n-1 & -1 & \dots & -1 \\ -1 & -1 & n-1 & \dots & -1 \\ -1 & -1 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & -1 \\ -1 & -1 & -1 & \dots & n-1 \end{vmatrix}.$$

(Note that this is the determinant of a matrix which looks exactly like M , but is an $(n - 1) \times (n - 1)$ matrix, while M is an $n \times n$ matrix.) Recall that to evaluate determinants in all but the simplest cases, it helps to make use of row and/or column operations: replacing a row/col with that row/col + multiples of other rows/columns (doesn't change the value of the determinant); replacing a row/column by c times that row/col (multiplies the value of the determinant by c); swapping two rows/cols (changes the sign of the determinant).

For this determinant, adding rows 2 to $(n - 1)$ to row 1 does not change its value, and produces the following determinant:

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ -1 & n-1 & -1 & \dots & -1 \\ -1 & -1 & n-1 & & -1 \\ \vdots & \vdots & & \ddots & \\ -1 & -1 & -1 & \dots & n-1 \end{vmatrix}.$$

Now add row 1 to each of rows 2 to $(n - 1)$, and we find that the number of spanning trees of K_n is

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & n & 0 & \dots & 0 \\ 0 & 0 & n & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \dots & n \end{vmatrix} = n^{n-2}.$$

By *Cayley's Theorem*, the number of labelled trees on n vertices is n^{n-2} .