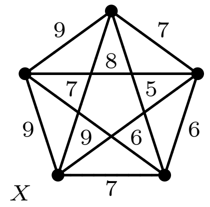
The University of Sydney

MATH2969/2069

Graph Theory **Tutorial 4 (Week 11) Solutions** 2008

1. (*i*) How many Hamiltonian cycles are there in this graph?



(*ii*) Delete the vertex labelled x (and its incident edges). How many spanning trees are there on the remaining subgraph? How many of these spanning trees are part of a Hamiltonian cycle?

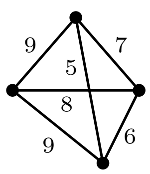
(*iii*) Find a minimum weight spanning tree on the remaining subgraph. Hence show that the weight (or length) of a solution to the travelling salesman problem is at least 32.

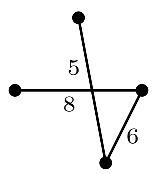
Solution.

(*i*) The graph is K_5, and so a Hamiltonian cycle is formed by starting at any vertex, taking the remaining 4 vertices in any order, and returning to the first vertex. Note that which vertex is considered as the initial (and final) vertex is not significant. For example, $v_1v_2v_3v_4v_5v_1$ and $v_2v_3v_4v_5v_1v_2$ are the same cycles. So, if the vertices are $v_1, \ldots, v_5$, we may without loss of generality assume v_1 is the start (and end). There are then 4! arrangements of the other 4 vertices. But each cycle has been counted twice in these 4! arrangements – once forwards, and once backwards. Therefore the number of different cycles is $\dfrac{4!}{2} = 12$.

(*ii*) The remaining subgraph is K_4. The number of spanning trees is equal to the number of labelled trees on 4 vertices, which is $4^2 = 16$, by Cayley's Theorem. There are 4 spanning trees on K_4 with degree sequence $(1, 1, 1, 3)$, and such trees cannot be part of a Hamiltonian cycle. The remaining 12 spanning trees have degree sequence $(1, 1, 2, 2)$ and are each part of some Hamiltonian cycle.

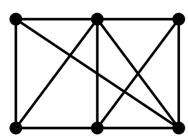
(*iii*) The graph which remains after deleting the vertex x and its incident edges, and a minimum weight spanning tree, are as shown:



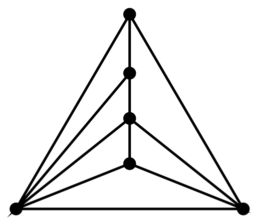


The minimum weight spanning tree has weight 19. The sum of the two smallest weights on edges incident with x is 13. Therefore $19 + 13 = 32$ is a lower bound for the weight of a solution to the travelling salesman problem.

2. Show that this graph is planar by drawing it in the plane without any edges crossing. Verify Euler's formula for this graph.



Solution.



Euler's Formula for planar graphs is

$$v - e + f = 2,$$

where v is the number of vertices, e is the number of edges and f is the number of faces. (Remember that f includes the exterior infinite face.) In this graph, we have $v = 6$, $e = 11$, $f = 7$, and so $v - e + f = 2$.

3. (*i*) Show that a connected simple planar graph all of whose vertices have degree at least 5 must have at least 12 vertices.

(*ii*) Show that a connected simple planar graph with fewer than 30 edges has at least one vertex of degree at most 4.

(*iii*) Show that a connected simple planar graph has at least one vertex of degree at most 5.

Solution.

In this question we use the following inequality, which holds for all simple planar graphs with 3 or more vertices:

$$e \leq 3v - 6.$$

The result is clearly true for all simple planar graphs with 3 vertices, and for all simple planar graphs with more than 3 vertices and 2 or fewer edges. For a simple planar graph with $v > 3$ and $e \geq 3$, each face is bounded by at least 3 edges and so $2e \geq 3f$. Therefore, using Euler's formula we have $3v - 3e + 3f = 6 \leq 3v - 2e + 3e$ and hence $e \leq 3v - 6$.

(*i*) If there are v vertices, all with degree at least 5, then the sum of the degrees is at least $5v$. But the sum of the degrees is twice the number of edges, and so we have $\frac{5v}{2} \leq e$. Using $e \leq 3v - 6$, we have

$$\frac{5v}{2} \leq e \leq 3v - 6, \qquad \text{or } v \geq 12.$$

That is, there are at least 12 vertices.

(*ii*) Suppose that every vertex has degree greater than or equal to 5. Then by part (*i*) $v \geq 12$, and $2e \geq 5v \geq 60$. That is, $e \geq 30$. So if $e < 30$, at least one vertex must have degree 4 or less.

(*iii*) Suppose that every vertex has degree greater than or equal to 6.
Then the sum of the degrees is greater than or equal to $6v$, and so $e \geq 3v$. This contradicts the result that $e \leq 3v - 6$. So we cannot have all the degrees greater than or equal to 6, and hence the graph must have at least one vertex with degree less than or equal to 5.

4. Consider a connected simple planar graph with v (≥ 3) vertices, e edges and f regions.

(*i*) Show that if $e = 3v - 6$ then each region is a triangle.

(*ii*) Deduce that a convex polyhedron with 12 vertices and 20 faces is composed entirely of triangles.

Solution.

(*i*) Note that the graph cannot be a tree, since otherwise we would have $e = 3v - 6 = v - 1$, or $2v = 5$. So each region must be bounded by at least 3 edges. Now, $v - e + f = 2$, so $3v - 6 = 3e - 3f$ and hence $3f = 2e$ if $e = 3v - 6$. If any region were bounded by more than 3 edges, then we would have $2e > 3f$. So if $2e = 3f$ each region is bounded by exactly 3 edges. That is, each region is a triangle.

(*ii*) Recall that a convex polyhedron can be considered as a simple planar graph (with $v \geq 3$, and each face bounded by at least 3 edges). So Euler's formula, $v - e + f = 2$, applies and if $v = 12$ and $f = 20$, then $e = v + f - 2 = 30$. But $3v - 6 = 30$, so we have $e = 3v - 6$, and hence each face is a triangle.

5. A graph is said to be *polyhedral* if it is simple, connected, and planar, and every vertex has degree at least 3.

(*i*) Prove that a polyhedral graph cannot have exactly seven edges.

(*ii*) Prove that no polyhedral graph has 30 edges and 11 regions.

Solution.

(*i*) Since the degree of every vertex is at least 3, we have $3v \leq 2e$, or $v \leq 2e/3$, and as before, $f \leq 2e/3$. So, if there were 7 edges,

$$v \leq \tfrac{14}{3} = 4\tfrac{2}{3} \quad \text{and} \quad f \leq 4\tfrac{2}{3}.$$

Since f and v are integers, this means $v \leq 4$ and $f \leq 4$.

Hence

$$v - e + f \leq 4 - 7 + 4 = 1,$$

which contradicts Euler's formula.

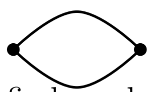
(*ii*) If there were such a graph, then, by Euler's formula,

$$v = e - f + 2 = 30 - 11 + 2 = 21.$$

Then $3v = 63 > 60 = 2e$, which contradicts the result $3v \leq 2e$. So there is no such graph.

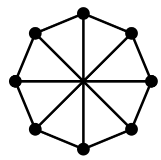
6. Give an example of a connected planar graph in which $e > 3v - 6$.

Solution.



$e = 2$, $v = 2$, $3v - 6 = 0$.

7. In this graph, find a subgraph which is homeomorphic to $K_{3,3}$.
Is this graph planar or nonplanar?

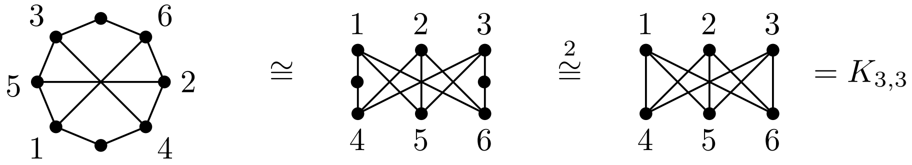


Solution.

Recall that two graphs are *homeomorphic* if they are isomorphic "up to vertices of degree 2", i.e., if an isomorphic copy of one can be obtained from the other by inserting a vertex of degree 2 into an existing edge, or removing a vertex of degree 2. Note that removing or inserting a vertex of degree 2 does not affect the planarity of a graph, so if a graph G has a subgraph homeomorphic to $K_{3,3}$ or to K_5, which are non-planar, then G must be non-planar.

To find a subgraph homeomorphic to $K_{3,3}$ (or to K_5), the usual technique is to remove carefully selected edges, leaving possibly some vertices with degree 2, but also leaving enough vertices of degree 3 (or 4) for the required homeomorphism.

Here, removal of any edge of the given graph will leave two vertices of degree 2, and six of degree 3 — a good chance of being homeomorphic to $K_{3,3}$, which has six vertices of degree 3. Let us remove, say, the edge from the top to the bottom vertex:



(where $\cong$ means 'is isomorphic to', and $\overset{2}{\cong}$ means 'is homeomorphic to').

It follows that the original graph is non-planar.